UNIVERSITY OF TAMPERE

APPEALING MULTIMODAL LANGUAGES TO ACCESS FIRST YEAR UNIVERSITY STUDENTS' UNDERSTANDING OF MATHEMATICAL CONCEPTS IN COSTA RICA

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HELEN ALFARO: Appealing multimodal languages to access first year University students' understanding of mathematical concepts in Costa Rica

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The current situation regarding the lack of skills and mathematical knowledge that students have when entering the university, has caused that institutions of higher education take certain actions such as the inclusion of courses or content reduction. Most of the measures taken involve curricular changes or partitioning of contents. However, the problem requires also methodological changes that improve students' understanding. Therefore, following the mathematical proficiency and the multimodal approach theories, this qualitative research seeks to use the written languaging exercises that involve the use of natural, symbolic and pictorial languages as a tool to address this situation, promoting the active participation of students to justify and explain their procedures. The aim is to find out student and teachers' experiences with the languaging exercises.

This research was conducted in a Calculus 1 course of the University of Costa Rica, with 33 engineering students and two teachers. The design involves three instruments to collect information: 17 exercises of languaging designed on the topic of derivatives that were applied during the class or as homework during seven weeks, a questionnaire with 18 Likert scale statements and six open ended questions answered by students after the applications of the exercises, and a semi-structured interview for the teachers.

The results indicated positive experiences of the participants. They expressed that the languaging exercises are useful to make learning more meaningful, to identify the different ways in which student's appropriate knowledge, as well as the misconceptions they have, through the explanations they provide. The exercises also favor, in their opinion, the development of analytical, reasoning, abstract thinking and metacognition skills.

Key words: languaging exercises, mathematical proficiency, university mathematics, knowledge gap.

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1 INTRODUCTION

The students' low performance in school mathematics has been extensively studied in recent years. The results of international tests, such as PISA and TIMMS, have generated a worldwide uncertainty about what and how to do in order to improve students' mathematical learning. It is known that the learning of mathematics has been historically difficult for the students in primary and secondary school and even at the beginning of university.

Previous studies show that the mathematics performance of high school students usually remains at the levels of reproducing procedures (Artigue, 1995; Valverde & Näslund-Hadley, 2011, Kilpatrick, Swafford & Findell, 2001; Programa Estado de la Educación, 2013), that is, applying algorithms or doing calculations. However, in higher level exercises (analyze, evaluate, and create) they do not know what to do. Students are repeating procedures; but, they do not understand the concepts and mathematical knowledge behind. Governments, researchers and schools have been doing changes and improvements in curriculums and teaching strategies with the aim to make the learning of mathematics more meaningful for students. However, the students' performance does not seem to improve.

The problem is not limited to the school. The participation of students in the first mathematics courses at the university is not satisfactory. Kajander and Lovric (2005) mention that "in spite of all efforts and energy ventured into the pre-tertiary mathematics education, the knowledge and skills of incoming university students are far from satisfactory" (p.149).

Hoyles, Newman and Noss, (2001) present three key areas that were causing problems to the undergraduate students, discussed in the report *Tackling the Mathematics Problems*, from the London Mathematics Society in 1995: one is the lack of fluency in the manipulation and simplification of numerical and algebraic procedures, second the low capacity of analysis, and third the fact that the students do not understand the importance of the logic and proof in mathematics, as a precise and exact discipline.

The transition from high school to university represents many changes in the lives of students: new routines, different peers, more independence, among others. But they also represent huge changes in the subjects in terms of the abstraction and complexity of the contents as well as the way in which they are addressed. School mathematics and university mathematics are very different.

As stated by Tall (1997), in the school the aim is to obtain an answer and; therefore, the students are taught to do computations and manipulate symbols. However, “at university there is a bifurcation between technical mathematics that follows this style (with increasingly sophisticated techniques) and formal mathematics, which seeks to place the theory on a systematic, axiomatic basis” (p.1). The mathematics taught in the university are more rigorous and abstract than the one studied in the school and besides, it acquires a formal approach that many of the students have never faced before (Luk, 2005).

Some researchers point out that the higher level of thinking at university mathematics, is other of the big changes in the transition (Gruenwald, Klymchuk & Jovanoski, 2004; Luk, 2005). The students have not been exposed enough to the mathematical thinking, but university professors see it as something natural, because “it is easy for mathematicians to take the mathematical way of thinking for granted, unaware that they may be talking in a foreign language to students” (Luk, 2005, p.163).

In addition, the discourse in the mathematics class in the university changes with respect to the classes in the school, as indicated by Barton and Neville-Barton (2004), “[l]ogical statements become the essence of mathematical meaning, not just a way of describing mathematical relationships. The roles of definitions, axioms and theorems in mathematical argumentation are subtly indicated in their linguistic expression” (p.14).

All of this changes joint with the lack of knowledge and mathematical skills of students when entering the university, create a gap in the transition from school to university mathematics. Because of that gap, universities have been facing problems with the students’ performance in the first mathematics courses (Hoyles et al., 2001; Daza, Makriyannis & Rovira, 2014; Kajander & Lovric, 2005; Sillius, 2011). The higher education institutions are expecting more from the students than students are prepared to offer. The concern about this gap is not new. Hoyles et al. (2001) mentioned that it was a common issue in the United Kingdom in 2001 and Gruenwald et al. (2005) stated that “many university lecturers feel that there is a need to investigate the ways of reducing the gap between the school and university mathematics” (p.12).

In order to address this problem, universities have carried out different actions. The European Society for Engineering Education (SEFI) mentions some of the measures taken by the universities. Among them we find the reduction of syllabus contents and the depth with which they are studied, the development of additional study units, the implementation of math centers of support or do nothing (Mustang &Lawson, 2002).

Similarly, Daza et al. (2014) point out that many "institutions now offer intensive courses prior to the academic year or even complete introductory courses running throughout the first term, whether in a classroom or (partially) online" (p.227). The University of Costa Rica, in 2015 introduced the pre-calculus course for students of engineering and economic sciences, who demonstrated serious deficiencies in mathematics in the diagnostic examination of knowledge and mathematical skills applied by the School of Mathematics, with the intention to fill the conceptual gaps of school mathematics knowledge of the students, which had been causing high rates of reprobation in the course of Calculus I.

Many of the actions taken to reduce the gap between school and university are oriented in modify the structure, order or amount of contents; however, less attention has been paid to the students' understanding of the mathematical concepts and the need to develop their mathematical thinking. White-Fredette (2009) refers to the fact that, in addition to the curricular modifications, changes must be made at the instructional level. For its part, Gruenwald et al. (2004) suggest that the teachers at university should find effective ways to help students to "understand the abstract concepts, master the formal language, follow rigorous reasoning, get a good feeling for the mathematical objects and acquire so-called mathematical maturity" (p.12).

With the intention of addressing the problem of the knowledge gap, but also considering the need to improve students' mathematical skills (Sillius et al., 2011; Rundgrén, Joutsenlahti & Mäkinen, 2016; Joutsenlahti & Kolju, 2017), some studies had been carried on in Tampere, Finland using the languaging approach. The studies introduced languaging exercises in the mathematics lectures, with the aim of help students to comprehend their mathematical ideas and thoughts and help them to achieve mathematical proficiency (Sillius et al., 2011). The use of different languages to represent concepts allows the student to understand better than if only one representation is used, promotes the conceptual understanding of mathematics (Chang, Cromley & Tran, 2016), and "form the basis for deep learning and fluency in working with mathematical ideas" (Wood et al., 2007, p.914). In the same way, the exercises favor the differences of learning of the students, offering them the opportunity to understand the mathematical concepts using the language that is more significant for them.

The purpose of this study is to utilize the languaging exercises in the University of Costa Rica, as an alternative option for reducing the knowledge gap and help students achieve their mathematical proficiency. Therefore, I designed and applied languaging exercises to first year student in the University of Costa Rica, through a design-based research method, to determine how students and their teachers, perceive the exercises and their usefulness in understanding mathematical concepts.

2 THEORETICAL FRAMEWORK

In this section, I present the theoretical foundations that guide this research. In the first instance, I discuss the situation of the study of mathematics that is experienced in the school and the levels of comprehension achieved by the students. Then, the mathematical proficiency framework is presented, which can be seen as a guide on the competences to be developed to improve the learning of school mathematics. Subsequently, the multimodal approach to mathematics is presented and some of the benefits of its use are discussed. Finally, the languaging approach developed by Joutsenlahti (2009) and colleagues is described as a way to apply the multimodal approach to achieve mathematical proficiency.

2.1 Mathematical skills of students entering university

In most educational processes, certain objectives are established and are expected to be achieved by the participants. In mathematics education, educational systems establish within their curriculums, knowledge and skills that students are expected to master satisfactorily. However, the results of investigations and evaluations carried out, show serious deficiencies.

Kilpatrick et al. (2001) argue that in the United States, school mathematics is highly related to the learning and practicing of computational procedures and that the "rules for manipulating symbols are being memorized but students are not connecting those rules to their conceptual understanding nor are they reasoning about the rules" (p. 234).

The International Development Bank (IDB) report on mathematics education in Latin America, points out the deficiencies of the education system and the worrying situation that is happening in the classrooms where the work is characterized by the memorization of routine computational operations and the mechanical reproduction of concepts (Valverde & Näslund-Hadley, 2011).

In Costa Rica, the last results of the PISA test, show that the Costa Rican students' capacities reach level one (Programa Estado de la Educación, 2013), which consist of answering questions located in familiar situations, with explicit and sufficient data; and developing routine procedures

(OECD, 2010). Students' math skills are fairly basic, they can perform procedures and solve equations, and some of them are able to make deductions, relationships, or interpretations.

According to this, the educational systems are favoring the learning of the mathematics centered in routine procedures that involve low level of conceptual understanding and reasoning, and the students are not reaching the level of performance expected. The students are leaving school with several deficiencies, they

> *do not understand the mathematical ideas which university teachers consider basic to their subject; they are not skillful in the manipulative processes of even elementary mathematics; they cannot grasp new ideas quickly or at all;(...) and, particularly, they have no sense of purpose-that is, they do not seem to realize that in order to study mathematics intensively they must work hard on their own trying to sort out ideas new and old, trying to solve test problems, and so on. (Thwaites, 1972, as cited in Hoyles et al., 2001, p.831)*

Students are not able to fulfill the expectations of the university. As Hoyles et al. (2001) stated, “They are therefore not prepared for the rigor and precision of university mathematics, and the requirement to make connections and abstractions rather than learn sets of recipes” (p.833). This panorama is not very encouraging. After going through all the school years, students are not being mathematically competent.

Some frameworks have been proposed to focus the teaching of mathematics on aspects that allow students to experience significant learning, as is the case of Mathematical Proficiency developed by Kilpatrick et al. (2001), which is presented below.

2.2 Mathematical Proficiency

There is always a need to improve theories, practices, and theoretical frameworks that guide teaching and learning practices. In mathematics education, research is conducted to find optimal practices that guide students to achieve a real understanding of mathematical knowledge and activity.

Kilpatrick et al. (2001) propose the mathematical proficiency approach after noticing that for a long time the learning of mathematics has been limited to knowledge and memorization without understanding. They suggest that if students achieve mathematical proficiency they will have the necessary components to achieve successful mathematics learning.

According to the authors, Mathematical Proficiency is what someone needs to be successful in the learning of mathematics. It is composed by five strands interwoven and interdependent: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

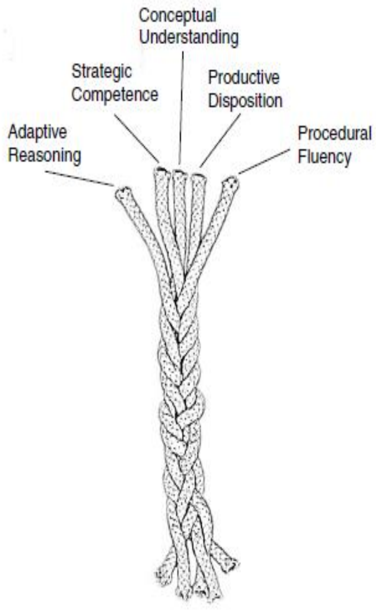


FIGURE 1. Intertwined Strands of Proficiency**.**

Note. Retrieved from "*Adding it up: Helping children learn mathematics",* Kilpatrick et al. (2001). Washington, D.C: National Academy Press.

- *Conceptual understanding:* It refers to an understanding of mathematical knowledge at such a level that can identify the connections between concepts and understand the justifications and reasons of the methods and operations. This understanding allows the student to remember and apply different concepts and methods correctly as well as monitoring their work. "Knowledge that has been learned with understanding provides the basis for generating new knowledge and for solving new and unfamiliar problems" (Kilpatrick et al., 2001, p. 119).
- *Procedural Fluency:* It "refers to the knowledge of procedures, knowledge of when and how to use them appropriately, and the skill in performing them flexibly, accurately, and efficiently" (Kilpatrick et al., 2001, p. 121). It is important to highlight the link of this component with the conceptual understanding, since otherwise the procedural resolution becomes a meaningless repetition of steps. Learning procedural fluency with understanding allows students to identify classes of problems that are solved with the same reasoning and achieve high

levels of skills that do not require extensive practice to master and can even practice on their own.

- *Strategic Competence:* It is related to the action of formulating, representing and solving mathematical problems, both within and outside the school context. It involves the knowledge of different ways of representing a problem mathematically, of different solution strategies and for which situations the knowledge is useful (Kilpatrick et al., 2001).
- *Adaptive Reasoning:* It is the capacity to think logically, considering the different relations between concepts and situations, as well as the consideration of the variety of alternatives and ways of reasoning. It involves the knowledge and the skills required to reflect and bring explanations and justifications for the steps in a procedure or for their own arguments. In order for a student to understand an algorithm, he must be able to explain and justify it, either for himself or for his classmates, as often as necessary. Kilpatrick et al. (2001) refers to adaptive reasoning as "the glue that holds everything together" (p. 129) and states that it includes formal proofs, different forms of deductive reasoning, informal explanations and justifications, and inductive and intuitive reasoning made by the identification of patterns, analogies and metaphors.
- *Productive Disposition:* It is related to the beliefs about mathematics. Students with productive disposition identify the utility and importance of learning and doing mathematics, identify how the mathematic is used to solve problems in the context and belief that being an effective learner and doer of mathematic is worthwhile (Kilpatrick, 2001).

The authors emphasize that it is important to develop and work with all the five strands in order to achieve mathematical proficiency, and that they should be considered both in the methodological strategies and in the tasks and exercises that the students must solve. However, like the studies mentioned in section 2.1, in the classroom the most practiced strand is the fluency of the procedures. Despite this, students fail to achieve adequate levels of performance in this area.

Getting students to be mathematical proficient involves encouraging them to reflect and be aware of their mathematical thinking, their understanding of knowledge, the connections between concepts and reasoning they do. That is, a higher level of understanding.

2.3 Multimodal Approach

Different studies had been done regarding the role that language plays in the mathematical practices, from different perspectives. One very important use of language in mathematic is for accessing those mathematical entities that are not considered concrete. Morgan et al. (2014) in their overview of language and communication in mathematics education, present two positions regarding the role of the language in mathematics. One of them, defended by Duval (2000, 2006), suggests that as we cannot access mathematical objects, because they are not palpable, the only way to do it is through the use of symbols, words, signs, expressions or drawings. From this perspective, we can notice that there are different modalities of language by which we can access mathematical objects.

Morgan et al. (2014) mention that in the mathematics education field, language is used in different ways: related only with words (e.g. natural language, verbal language) or using non-verbal modes of communication (includes mathematical symbols, diagrams, graphs, gestures). The authors highlight the importance to recognize and address the multimodal nature of the mathematical communication, and suggest the description and studying of the different modalities (Morgan et al., 2014). Likewise, O'Halloran (2015) reinforces the multimodal characteristic of mathematic language, and states that

> *the multimodal (or multisemiotic) makeup of mathematics means that three different meaning potentials are accessed to construct mathematical reality: namely, linguistic, symbolic and visual forms of representation, each of which have developed specific grammatical features to fulfill the functions they are required to serve. That is, language is used to reason about the mathematical results in a discourse of argumentation in which mathematical processes are related to each other and interpreted. (p.71)*

Each of these languages has associated characteristics, functions and grammatical difficulties that are detailed below.

2.3.1 Natural Language

The natural language used in mathematics is usually applied in the production of explanations, argumentations and discourses; in other words, "to create a discourse that moves forward by logical and coherent steps, each building on what has gone before" (Halliday, 1993, p. 64). As stated by O' Halloran (2015), this language "operates to foreground and background concepts which are related to each other through technical taxonomies and relational processes to form chains of reasoning" (p.71).

When using the spoken language in mathematics, we can use words that only have meaning inside the mathematic context, but also use common words that acquire a specialized meaning in the mathematic field. Besides, there are differences between the natural language used by the mathematicians and the one used in the classrooms (Morgan et al., 2014).

Halliday (1993), present a set of difficulties associated with the use the natural language that include the relations between concepts and definitions across the theory, the technical taxonomies, terms specific to mathematics, the large amount of information that can be implicit in a statement (definitions, theorems), among others.

2.3.2 Symbolic Language

According to O'Halloran (2015), this language emerges as a source to address mathematical entities that cannot be addressed by the spoken language. Its actual standardize structure is the result of an evolution process during history, influenced by political and social aspects. She references that the mathematical symbolism

> *is used to capture relations between mathematical entities and processes and derive results through a grammatical organization which retains participant and process configurations through the use of special symbols, specific conventions and deep levels of embedding. Meaning is encoded economically and unambiguously, resulting in a robust, flexible tool for reasoning about mathematical reality in a congruent, dynamic form. (p.71)*

The fact that this language allows to summarize the information and present it in a precise way that does not present ambiguity is stressed by the author, and liked by many mathematicians and even mathematics teachers. Driver and Powell (2015) identify that the tendency in classrooms is to present and solve the mathematics problems almost exclusively using symbolic language; however, this trend, causes difficulties in students. O' Halloran (2015) presents some difficulties regarding the use of the symbolic notation, in the following chart:

TABLE 1. Grammatical difficulties in mathematical symbolic notation.

Grammatical difficulty		Explanation and Examples
(a)	*Special symbols*	Conventionalized use of special symbols which combine in precise ways: e.g. $X(\alpha)=\alpha_1 X_i+\alpha_2 X_j+\alpha_3 X_k+\alpha_4 X_l$
(b)	*New grammatical strategies*	Grammatical strategies for encoding meaning differ from language: e.g. use of spatial notation for indices, rule of order for operations, ellipsis of operations and use of brackets.
(c)	*Density of symbolic configurations*	Content information is packaged into special symbols and embedded configurations of mathematical processes and participants: i.e. all symbols have content meaning
(d)	*New process types*	Mathematical processes differ semantically from those found in language and unfold according to rules of order: e.g. $f(x,y,z)=x^2+y^2+z^2$
(e)	*Chains of implicit reasoning*	Implicit conjunctions which underpin semantic continuity are often based on prior knowledge: i.e. the basis for logical meaning is not always explicit.
(f)	*Recoding of uncertainty*	Statements are typically non-modalized e.g. probability statements are used to encode uncertainty.
(g)	*Decontextualized knowledge*	Abstract, generalized participants and processes are contextualized in relation to other each other: e.g. mathematical statements are not context dependent with respect to the situational environment.

Note. Retrieved from "The language of learning mathematics: A multimodal perspective " by O' Halloran, 2015, *The Journal of Mathematical Behavior*, 40, p. 69.

As we can observe in table 1, the difficulties include problems of interpretation of the concepts behind the symbols, as well as confusions about the functions of certain symbols that differ from how they are used in the natural language. Difficulty in interpreting or translating mathematical symbols, following the idea in a solution or expressing the answers in symbols, affects the students learning (Sillius et al., p. 431).

2.3.3 Pictorial Language

The pictorial language, together with the symbolic one, allow in areas of mathematics such as analytical geometry, to formalize and visualize mathematical entities and processes. Likewise, through the images "mathematical relations are visualized, opening up a vast potential for viewing the mathematical representation as a whole and the parts in relation to each other" (O'Halloran, 2015, p. 71). Images like charts, function graphs or geometrical figures, can include and summarize a great amount of information, relations and characteristics of mathematical objects.

However, the use of this language in mathematical proof or work has been devaluated. Presenting the results in symbolic language has been more accepted in the academic field. Nevertheless, in recent years, technological advances have returned some credibility to this type of language, offering images of very complex entities, in order to make possible their study (O'Halloran, 2015). Regarding the difficulties in the use the pictorial language, O'Halloran includes the dealing with special conventions, the density of visual information, the implicit reasoning and the embedding of symbolic and linguistic elements.

After exposing the importance, functions and characteristics of the three languages it is evident that in the classrooms one must work on the literacy and integration of the three languages (O'Halloran, 2015) for a better understanding of mathematical meanings and concepts.

The languaging approach developed by Joutsenlahti (2009) considers the multimodal characteristic of mathematical language, focusing on written exercises. The author, together with other researchers, implemented studies to observe students' experiences with exercises that involve different languages of mathematical language.

2.4 Languaging

As an alternative to promote the strands of mathematical proficiency using the different languages presented in the multimodal approach, Joutsenlahti and a group of researchers of Finland have developed the languaging approach. Languaging can be understood as the student's expression of his/her mathematical thinking using different languages (Rundgrén et al., 2016). Joutsenlahti (2009), considers three languages in his researches: mathematical symbolic language (SL), natural language (NL) and pictorial language (PL).

In Finland, some studies have been carried out using the languaging in mathematics as a method to improve the learning of the students. For example, Rundgrén et al. (2016) designed languaging exercises in structural mechanics in order to find out how students experience this kind of exercises. In a very different context, Joutsenlahti and Kolju (2017) developed a study with fourth grade pupils, using languaging approach in order to study how students understand the concept of division.

Through the different researches carried out, Joutsenlahti (2009) identifies five different models of languaging exercises: standard model, storytelling model, roadmap model, comment model and diary model, detailed below.

The standard model is presented in the exercise where the statement and the solution are given using only symbolic language, and is one of the most found in school books and classrooms (Driven & Power, 2015; Ojanen, 2016). The procedure consists of some computations that will lead to the answer, which gives very little possibility to interpret the understanding of the concepts.

The storytelling model involves symbolic and normal/pictorial language. Languages take turns allowing the solver to explain or clarify their procedures with words or images, which makes it later easier to read and understand (Ojanen, 2016).

The road map model begins with a statement in natural or pictorial language, followed by calculations or procedures in symbolic language, to conclude with a response. The majority of the cases the answer is in natural language and the initial question is related to the result.

The comment model has a structure similar to a table with two columns, one column shows the steps or calculations in symbolic language, and the other justifications or comments (Ojanen, 2016). This model allows the reader to understand why the calculations are carried out and sometimes the rules or theories that support them.

In *the diary model*, the solver uses the different languages as it deems convenient or better. It can be full of explanatory comments or drawings about calculations or procedures.

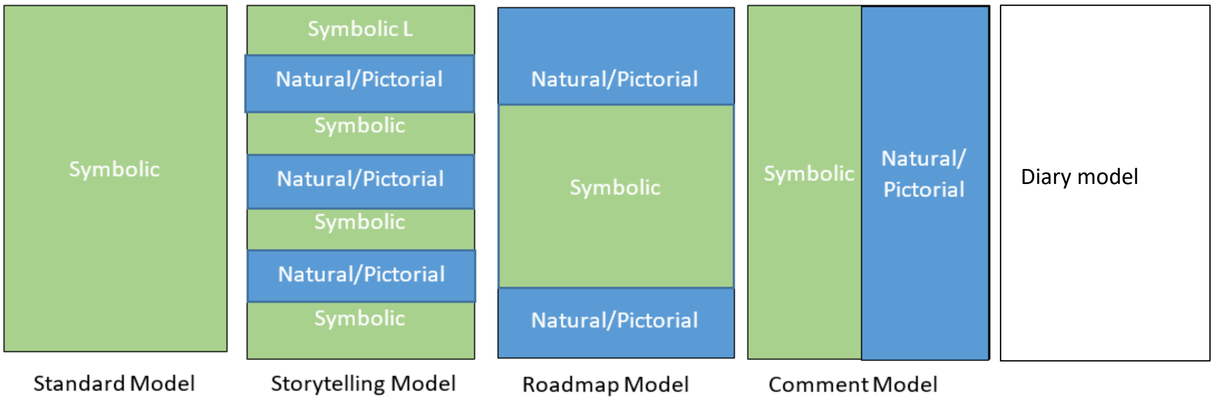


FIGURE 2. Languaging exercises models

In later studies, it was possible to identify and characterize eight kinds of exercises: code-switching, adding missing parts, from solution to word problem, seeking errors, argumentation of the solution, data filtering, and explaining in your own words and organizing (Rundgrén et al., 2016). These types of exercises can be mixed between themselves, and engage in the same task, which may have some of the models presented previously.

The languaging exercises are written exercises. It is known that students are accustomed to writing a lot in mathematics. In their writings, however, what predominates are long procedures and calculations in symbolic language (Morgan, 2002). Written languaging exercises promote that, in addition to symbols, students use natural and pictorial language to help them in the meaning making process. Through writing it is possible to exteriorize the students' thinking, and when a student is engaged in the solution of a written assignment, he/she should try to be as clear and concrete as possible, so that the answer will be understood by the reader (Morgan, 2002).

The action of writing justifications and explanations, requires that the students while ordering their thoughts, have to review and clarify to themselves the mental processes they did before explaining it to others or putting it on paper. This will enhance the student's understanding (Kline & Ishii, 2008; Sillius et al., 2011).

Besides, having the written material of the students make easier for the teacher to identify that the he/she has incorporated the new concepts and what meaning the pupil has given to it. For students, having the written records permits to go back to them later and understand what was done in the exercises (Morgan, 2002; Sillius et al., 2011).

3 RESEARCH QUESTIONS

Considering the arguments presented about the knowledge and skills gap with which students enter the first mathematics courses in university, and based on the theory presented in the previous chapter, this research aims to answer the following questions:

1. **How do the students express their thoughts?**
    1.1. How did the languaging exercises evidence the Kilpatrick's et al. (2001) features of the model?
2. **How do students perceive languaging exercises?**
    2.1. What difficulties did the students face while solving languaging exercises?
    2.2. What advantages did the students find in using languaging exercises themselves?
    2.3. What disadvantages did the students find in using languaging exercises themselves?
3. **How do the teachers perceive languaging exercises?**
    3.1. What advantages did the teachers identify in using languaging exercises?
    3.2. What disadvantages did the teachers identify in using languaging exercises?
    3.3. How did languaging exercises support their teaching?

The question number 1 was answered through students' responses to languaging exercises. The answers of the questionnaire solve the question 2, considering both the Likert scale and the open ended questions. The interviews with the teachers gave the information to answer question number 3.
In the next chapter I will describe the methodology followed in order to answer the research questions.

4 RESEARCH DESIGN

In this chapter, I present a description of the research design, application and analysis. First, I introduce the approach that guides this research, as well as the methodology. Second, the design processes of the materials, the materials as such and the application process are described. Thirdly, I present a description of the participants in the research. Finally, I explained the strategies used to analyze the data collected.

4.1 Methodological framework

This study corresponds to a Design-based research. This methodological approach, which had its initial stages mainly at the beginning of the 21st century, has been welcomed by researchers in the field of education, and several studies have carried out it in the lasts years in different disciplines, countries and educational levels (Anderson & Shattuck, 2012.) The Design-based research emerged as an alternative to solve the recurrent criticism of linking theory to practice. Brown (1992), who is considered as the pioneer of this research approach, thought that the challenge "was to develop a methodology of experimenting with intervention designs in situ to develop theories of learning (and teaching) that accounted for the multiple interactions of people acting in a complex social setting" (Sandoval & Bell, 2004, p.199).

Therefore, as mentioned by Anderson and Shattuck (2012), in their article "Design-Based Research: A Decade of Progress in Education Research, the design-based research can be understood as

> *a methodology designed by and for educators that seeks to increase the impact, transfer, and translation of education research into improved practice. In addition, it stresses the need for theory building and the development of design principles that guide, inform, and improve both practice and research in educational contexts. (p.16)*

In this approach, theories or theory driven hypothesis or assumptions are considered, to design interventions in real school contexts. Bell (2004) highlights that this approach considers "that we can learn important things about the nature and conditions of learning by attempting to engineer and sustain educational innovation in everyday settings” (p.243). Being situated in a real educational context is in fact one of the features that define this approach, according to Anderson and Shattuck (2012), it gives validity to the study and allows to evaluate, inform and improve the practice.

The authors also consider that the designs made with this methodology must have the characteristic of being applicable by teachers and students in different contexts and with different characteristics. They mention that at the time of designing, it should be consider that the intervention is feasible in terms of time, complexity and instruments.

The use of mixing methods and several instruments to collect data is common in design-based research, since, as pointed out by Maxcy (2003) “It is perfectly logical for researchers to select and use differing methods, selecting them as they see the need, applying their findings to a reality that is both plural and unknown” (p. 59). It means considering the characteristics of the problem and the context, the researcher can design several instruments, so that the intervention is as successful as possible.

Another important feature discussed by Anderson and Shattuck (2012) is the cooperative work between researchers and practitioners. They consider that the researcher is the one that has domain over the theoretical assumptions and research practices, but the teacher has the knowledge of the needs of the context, educational policies and the participant subjects.

Likewise, the design-based research is characterized to have an impact in the practice, and it aims to design studies in order to understand and improve some learning process (Coob, 2001). Through the interventions, the principles that guided the design are challenged, improved, and at some point, after several iterations with the findings it will be possible to create a generalizable theory. In other words, “Designs evolve from and lead to the development of practical design principles, patterns, and/or grounded theorizing” (Anderson & Shattuck, 2012, p.17).

According to Edelson (2002), the research approach is very useful to improve the education system because the problems that motivate it are taken from the educational context and the results it offers (activities, materials, systems…) are directly applicable in the school context.

Due to the nature of this approach, the research carried out follows an interpretive paradigm, since it is intended to perceive and understand the experiences of different individuals when performing languaging exercises. To collect the data, I used quantitative methods, such as the questionnaire with the Likert scale, and also qualitative methods, like open-ended questions, interviews and solutions for languaging exercises. Therefore, the greatest weight of the research is qualitative.

4.2 Data collection

Different instruments were used to collect the information. First, the languaging exercises designed according to the contents corresponding to derivatives present in the course program (see Appendix 1). Then the questionnaire with 18 Likert scale questions and six open-ended questions was used to know the opinion of the students regarding the languaging exercises. Besides, a semi-structured interview was conducted with the two teachers in charge of the groups to compile their opinions and recommendations regarding the exercises and the experience of using them in the class. I compiled the information from these three sources to have a complete overview of the participants' experience, of using the languaging exercises to study derivate in the course of Calculus I.

Since all the data was collected in Spanish, the excerpts from the interviews and the answers to the open-ended questions presented in chapter 5 were translated by the researcher.

4.2.1 Design of Languaging exercises

Considering the studies carried out by Joutsenlahti and colleagues in Finland, including the models and types of languaging exercises described in chapter 2, I proceeded to design specific exercises for the program of the Calculus I course, offered by the School of Mathematics of the University of Costa Rica, for Engineering students. The contents used to design the exercises were the knowledge about derivatives, which would be evaluated in the second exam, according to the course syllabus.

Following the course timetable, I analyzed the contents that were supposed to be studied each week and based on my experience of teaching that course, I selected the most relevant contents considering the importance of the topic and the utility of the use of languaging. The purpose was to design approximately two exercises per week.

In total, 17 exercises of languaging were designed, using the models and types of exercises identified by Joutsenlahti and colleagues. There were exercises of completing missing steps and justifying procedures, sorting steps and identifying the rules and properties involved, identifying errors and explaining the correct way to solve them, interpreting instructions given in natural language and using them to solve the exercise in symbolic language, giving examples using symbolic, natural and pictorial language, interpreting information from a graph and giving explanations, explaining possible route for solving a problem using only natural language, among others.

TABLE 2. Description of the languaging exercises

#	Topic	Structure of the Exercise
1	Definition of derivate using limits	Complete missing steps given in SL and explain with their own words the rules and procedures used, with NL
2	Relationship between continuity and derivability	Explain with NL the procedures they will use to solve an exercise, providing justifications, without using symbolic language
3	Possible cases where a function is not derivable	Represent a situation using the three languages: SN, NL and PL, completing a chart
4	Rules of derivation of algebraic and trigonometric functions	Identify mistakes in a given solution in SL and correct them. Provide justifications for the solution steps, with NL
5	Tangent line to a curve	Order given steps in NL and perform the respective calculations, using SL
6	Implicit derivation	Order given steps in SL and comment what happens in each stem, with NL
7	Rate of change problems	Explain a given solution in SL, to be understood for any of the students in the class, with NL
8	Logarithmic derivation rules	Follow the instructions given in NL to solve an exercise using SL
9	Derivation of logarithms	Complete missing steps in a solution given in SL and explain what happen in each step, with NL
10	Logarithmic derivation	Order the solution steps given in SL and justify them using NL
11	Derivate of the inverse trigonometric functions	Justify with NL the steps given in SL and complete them, if they believe is necessary
12	Absolute and relative extremes. Critical point	Evaluate two given opinions and provide the best solution for the situation. There is a graph to use as reference

13	Calculation of extreme values for a continuous function in a closed interval	Explain in their own words, the need and justification of given steps in NL, in order to solve an exercise
14	Complete study of a function given its criteria: domain, intersections with axes, asymptotes, critical points, classification of relative extremes, growth and decrement intervals, inflection points, concavity, summary and plot	Explain in their own words using NL the graphic implications of some statements in SL and use them to provide a sketch of a graph that meets the conditions given with PL
15	Relationship between the monotony of a function and the sign of the first derivative. Relationship between the concavity of a function and the sign of the second derivative. Inflection point	Interpret information about the function from a graph of the derivate, PL, and justify statements, with NL
16	Optimization Problems	Write in their own words, using NL, the explanations they will give to solve a problem, with the pertinent justifications
17	Derivation Rules	Identify mistakes in three different given solutions and provide the correct answer

Acronyms: Symbolic language (SL), natural language (NL) and pictorial language (PL)

The exercises were designed to allow students to experience the use of different languages while studying derivate, considering the strands of mathematical proficiency, especially procedural proficiency, conceptual understanding and adapting reasoning.

These three branches are closely related. The presence of procedural fluency can be evidenced in several exercises: for example, the exercises of completing missing steps, ordering steps of a procedure or following instructions given to perform the procedures of solving an exercise. Many of these tasks require students to make procedural justifications, either by identifying the formulas or properties applied or properly arguing the reason for these procedures, which implies that they have sufficient conceptual understanding of the contents involved.

In the exercises in which they must provide explanations of the procedures that they would follow to solve an exercise or in which they must identify errors, the students must access to the theory to be able to realize it. The conceptual understanding is thus evidenced.

The adaptive reasoning is also present in several exercises. For example, the students should monitor the solutions of the exercises to be able to identify the errors. They must make connections between knowledge to interpret the information of the function from the graph of its first derivative and in general to provide justifications.

The strategic competence was attempted to include in designing exercises that were different from what students are accustomed to solve, such as in exercise 12 (appendix 13) where they should evaluate two opinions regarding how to solve a problem or exercise 15 (appendix 16) in which they were to interpret information about the function f, from the graph of the first derivative. It is difficult, however, to guarantee that they were new to all students, or that the course teacher had not previously introduced them.

The strand of productive disposition is difficult to evidence in the exercises since the motivation on the learning and use of mathematics is an attitude fed in an individual way. However, including exercises such as exercise 12, which is related to an application of mathematics in real life, or exercises 11 (appendix 12) and 8 (appendix 9) in which they probe some of the formulas they use to derivate, the results might be that the students find more sense in learning mathematics and increase their motivation.

The languaging exercises were developed at the University of Tampere, Finland, in Spanish, then the researcher translated them into English. The exercises were reviewed by several colleagues and by the two teachers who were going to apply them. After including the feedbacks, they were sent by email to Costa Rica for the application. The electronic mail was the main means of communication between the application teachers and the researcher, during the application of the exercises.

4.2.2 Implementation of the exercises

Regarding the implementation of the languaging exercises, teachers were completely free to decide at what time in the class they will like to apply the exercises and how to do it. However, they were given suggestions in terms of the content for which each exercise was designed, the role they should assume as teachers and the role of the students.

One of the professors decided to use the exercises in the class, giving students enough time to solve them and after that comment them. However, in some cases the time in class was not enough, so the exercises were given as homework. On the other hand, the teacher of the second group decided to use all the exercises as homework, and used the office hours to comment them.

After each session of exercises, the professor picked up the exercise sheets and the students have the opportunity to take pictures if they wanted to keep their answers. Subsequently the application of the 17 exercises, the students were asked to answer a questionnaire about their experience with the languaging exercises.

4.2.3 Questionnaire

The questionnaire applied has two parts. A Likert scale with 18 statements, that is defined by Nemoto and Beglar (2014) as "a psychometric scale that has multiple categories from which respondents choose to indicate their opinions, attitudes, or feelings about a particular issue" (p.2) and six open-ended questions.

The statements of the Likert scale had four levels of agreement: strongly disagree, disagree, agree and strongly agree, following the advice of Nemoto and Beglar (2014), who suggest that for a young population that may have less motivation to answer the questionnaire, it is more convenient to use only four response options. In the same way, the authors emphasize the importance of not having a neutral option response, since it can obstruct the analysis of the information and does not represent a strong opinion on the subject.

The statements 1-16 were taken from Sarikka (2014). These statements were in Finnish, so they were translated into English and revised by two mathematics teachers whose mother tongue is Finnish, with the purpose of preserving the desired intention. The last two statements were designed by the researcher in order to complement the information needed according with the research questions. The statements refer to students' opinion about their abilities in mathematics, how useful they think languaging is for solving and understanding exercises, explaining to others or understanding what others did.

The second part includes six open-ended questions concerning the students' opinion about the use of natural language and pictorial language, and their benefits. Also they were asked to comment on their experience of using languaging exercises, for what they can use languaging in their studies and how the exercises helped in the development of the class.

The researcher asked the students to be as honest as possible when filling out the questionnaire, since their answers would be used to evaluate the languaging exercises.

4.2.4 Teachers' Interviews

After the application of the exercises and the questionnaire, the two teachers in charge of the participants' groups took part in an interview with the researcher. The interviews were conducted individually, had a semi-structured design and were carried out during the office hours of the teachers. The aim of the interviews was to know teachers' opinion regarding the advantages and disadvantages of using languaging exercises in the class, as well as the impact that languaging exercises could have in students' understanding of contents. The interview was divided into three main themes: how language exercises benefit the students when studying, the usefulness of these exercises for teaching, and finally their opinion on the exercises as such. The interview was recorded and the researcher took support notes during the meeting.

4.3 Participants

The languaging exercises were applied in two different groups of Calculus I, in the University of Costa Rica, in the first semester of 2017. This course is offered to engineering students, who must have passed the course of pre-calculus or have demonstrated enough previous mathematical knowledge to be able to take it.

The participation in the study was voluntary. For that reason, although the population of the two groups was approximately 60 students, it was possible to collect a significant sample of exercises (at least 11 out of 17) from only 33 students. The students were asked to solve 17 languaging exercises related to the contents they were studying on the topic of derivatives. After all the exercises were done, they answered a questionnaire with 18 Likert scale questions and six open-ended questions, regarding the experience of using languaging exercises. Participants were informed that the collected data was going to be used anonymously and that their performance in the exercises would not influence their grades.

The two teachers responsible of the groups also collaborated in the research, through a semi-structured interview. Both teachers have a Licenciature degree of the University of Costa Rica in Teaching of Mathematics, and they have taught the Calculus I course several times.

The participants were chosen for several reasons. First, the students were taking a first mathematical university course, which coincides with courses where knowledge gaps are evidenced. Second, since languaging exercises require justifications and explanations, it was more convenient to use subjects with the same native language as the researcher, in order to avoid language bias. Third, the

Mathematics School of the University of Costa Rica and the teachers in charge of the groups offered facilities of access and organization.

4.4 Data analysis

The analysis was informed by different sources and was carried out through four stages. The first stage corresponds to the quantitative analysis of the Likert scale results. After that, a qualitative content analysis was carried out for each of these data: the answers to the open-ended questions of the questionnaire, the teachers' interviews and the solution of the exercises. Figure 3 offers a visualization of the triangulation process.

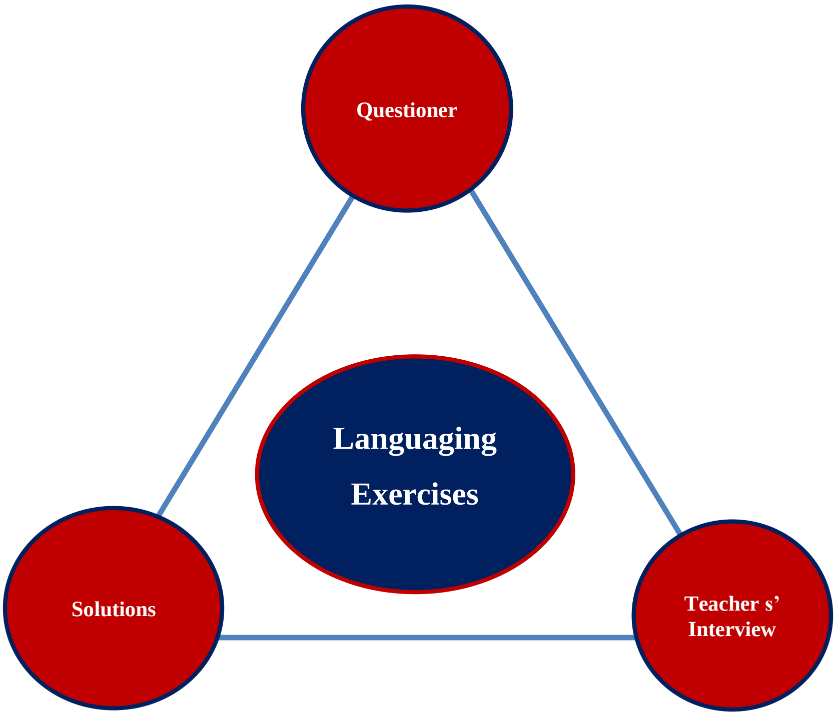


FIGURE 3. Analysis design

4.4.1 Quantitative analysis of the Likert scale

From the quantitative analysis of the responses of the Likert scale, it is intended to obtain insights of students' opinions about their abilities for mathematics, the benefits and difficulties of using different languages in the approach and solution of exercises, as well as in explanations and justifications. To carry out this analysis, the calculation of the absolute and relative frequencies of each agreement level was made for each statement. However, in order to make the conclusions, the cumulative frequency of the students who had answered "agree” or “strongly agree" was considered. The mean and standard deviation of each question were also calculated to observe the behavior of the answers with respect to the mean.

4.4.2 Qualitative content analysis of the open-ended questions

The qualitative content analysis that was carried out with the answers to the open-ended questions, sought to complement the insights obtained from the results of the Likert scale. The first step that was carried out for this analysis was a first reading of the 33 participants’ answers of the six open-ended questions. From this initial reading, all the topics mentioned by the students were extracted, counting the ones mentioned more than once. The next step was a first attempt to establish categories to locate the topics found. This attempt generated eight categories. After this, I proceeded to construct the definition of the categories, to delimit the characteristics of the topics included in them, which reduced the categories to six, two of which had three subcategories.

With the established categories, I reread the answers, placing each of the subjects in the new categories and counting the number of times they were mentioned by the students, which produced some changes to the initial classification. From this second reading, I decided to merge two categories referring to the benefits of languaging exercises, so that this category was left with six subcategories. The final categories are: Lesson development, difficulties, benefits of using different languages, possible uses and general perception of the experience. They are presented in tables to make it easier to read with the topics and the number of times mentioned in the answers.

Considering the categories, interpretations and contrasts with the results of the Likert scale were done, in order to have a broader view of the opinions of the students.

4.4.3 Qualitative content analysis of the teachers’ interviews

The interviews of the professors, are intended to inform about their opinions on the languaging exercises for the learning of the students, and for the development of the class. The first step for this qualitative content analysis was the transcription of the interviews. The interview of the first professor, P1, lasted 48 minutes and the one of the second professor, P2, 18 minutes, approximately. When the transcription was done, I proceed to read it and interpret if the resultant categories of the qualitative analysis of the open-ended questions were suitable for the main topics discussed in the interviews. However, the topics had different perspectives. Hence, after reading them for the second time, I decided to organize the information in three categories: aspects related to the benefits of using languaging exercises for students' learning and understanding, the disadvantages they found and the features related to the teaching practice. Considering the previous step, I read the information again and locate the topics according to the established categories.

4.4.4. Qualitative content analysis of the languaging exercises

The fourth stage was about the analysis of the answers to the languaging exercises. I decided to analyze the solutions of only 17 of the 33 participants, for two reasons. First those students completed the solutions of the 17 languaging exercises and second, the amount of information was quite dense. The aim of this analysis was to observe how students expressed their thoughts, and how the answers evidenced the strands of mathematical proficiency.

This step of analysis was the most complex of all. Like the previous stages, the first step was to read all the answers of each exercise, and to extract the most important and repeated elements. I considered the content knowledge they demonstrated, conceptual and procedural errors, the type of language that dominated the answers, and for what they used it, or if there was a combination of several languages, among others. However, the information was too broad to identify common or general topics. Therefore, I proceed to group the exercises according to their characteristics and objectives, so that I could focus the analysis on the main topics according to the type of exercise. As a result, I got five groups of exercises. After that, I reread the solutions to the exercises again but focused on the exercises characteristics and the coincidences among them.

With the insights of the second reading, it was possible to recognize some dominant aspects in each group. For example, in group A the topics were distributed in three categories: the use of natural language to refer to procedures, operations or properties; different uses of languages such as translation of the properties from symbolic language to natural language, or combination of them; and some knowledge aspects like notions that students have about concepts, as well as observing that in some cases students recognized the intentions behind certain procedures. Nevertheless, this categorization was different for each group. After identifying the main topics of each group of exercises, I made the analysis of the main features, using images and extracts of the answers to illustrate the statements.

The sections described offer an overview of the process carried out during this investigation, from the design of the materials to the implementation. The purpose of this process is to achieve the aim of this research: know the opinion of the students and the participating teachers about the languaging exercises and their usefulness to improve learning. In the next chapter, the description and analysis of the results are presented.

5 RESULTS AND DISCUSSION

The purpose of this chapter is to present the results obtained after applying the instruments for data collection and the analysis described in chapter 4. The chapter has three main sections, which derive from the three instruments used. First, the analysis of the solutions of the languaging exercises is presented, complemented with examples of the work of the students. Second, the answers to the questionnaire are analyzed, considering the Likert scale and the open-ended questions. The third section corresponds to the analysis of the interviews carried out with the two teachers in charge of the groups.

It is important to mention that languaging exercises designed, are also part of the results of this research. The exercises represent an application of the multimodal approach to improve student learning about derivatives, and can be found in the appendixes.

5.1 Students' understanding of mathematical concepts and processes evidenced in the exercises

In order to analyze the different evidences of students´ understanding about the topic of Derivate through the languaging exercises, the 17 exercises were classified in five groups, according to their characteristics. In group A are the exercises 1, 4, 6, 9, 10, 11 and 17. These involve ordering or completing steps, as well as identifying errors in procedures given in symbolic language. All these tasks are accompanied by providing justifications or descriptions of the properties or rules used in each step. Group B includes exercises 5 and 8, which consist of solving the steps that are given in natural language, using symbolic language. The exercises that demand an explanation of how to solve a problem, are 2,7 and 16 and are in group C. Group D includes exercise 3 in which students must use all three languages to complete a table. Finally, group E contains exercises 12, 13, 14 and 15, which require students to interpret or evaluate affirmations given in symbolic, natural and pictorial language and offer explanations of the validity, necessity or utility of those statements in the procedures involved. In the following sections, I present the analysis of each group of exercises.

Group A

The exercises of this group (1, 4, 6, 9, 10, 11 and 17) allowed to observe aspects denoting the use of natural language to refer to procedures, operations or properties. They also revealed different uses of language such as translation of the properties from symbolic language to natural language, or combination of them. With regard to knowledge, it was possible to identify some notions that students have about concepts, for example about inverse function; as well as observing that in some cases students recognized the intentions behind certain procedures.

Concerning the use of natural language, there were six examples where students refer in different ways to the same procedure. One of them was about the use of the trigonometric identity of the cosine of a sum of angles (Appendix 2), i.e. $cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$. Students used different verbs to indicate that the identity was used, as well as different "names" for the cosine of a sum of angles. Some students preferred to use symbolic rather than natural language to refer to the identity.

Do the cosine of a sum (S17)

Use the formula of addition of angles (S18)

Develop the sum cosine (S19 & S20)

A sum of variables was developed within the cosine (S24)

Apply the trigonometric property of the sum of cosine angles (S21, S25 & S26)

Solve the cosine of a sum (S22)

Develop the cosine formula of two added functions (S33)

Develop angles addition in $cos(x+h)$ (S32)

Apply the identity $cos(x+h)=cos\,x\,cosh-sin\,x\,sin\,h$ (S31)

Solve $cos(x+h)$ by $cos\,x\,cosh-sin\,x\,sin\,h$ (S30)

The quotations show that from 17 solutions considered, students used 13 different ways to justify the same step. Some of the notions are not correct or can lead to mistakes, as for example the interpretation of student S24 may lead to a misunderstanding adding the terms within the argument of the cosine, and that is not what the identity states. However, it is relevant to mention that almost all students used the identity in a correct way. There were only three cases with an error of signs and another that used the identity of the sine of a sum of angles instead.

There were also different ways students translated logarithmic properties to natural language, as shown in Table 3.

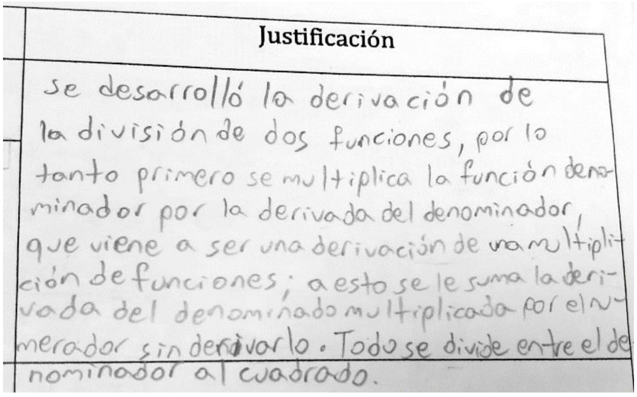
TABLE 3. Students' translations of logarithmic properties

$\log_a x^n = n \cdot \log_a x$	$\log_a \frac{x}{y} = \log_a x - \log_a y$	$\log_a x \cdot y = \log_a x + \log_a y$
If a logarithm is raised to an exponent, this [the exponent] can be multiplied to the front. (S18)	Separating a division as the subtraction of two logarithms. (S18)	Separate a multiplication as the sum of two logarithms. (S18)
The exponent of the criterion goes to multiply. (S20)	The division is written as subtraction (S20, S21, S24 &S33)	
Take the exponent to multiply the logarithm. (S33)		

In this case, in the exercises, the students' task was to recognize the property that was being applied, and therefore, these translations are understandable if you have as reference the step in symbolic language. Nevertheless, they lack important details such as that the "division property" applies to the logarithm of a division, but not to the division of logarithms.

Phrases like "change signs" or "split the -1" were identified to refer to the action of distributing a -1; "move to divide" or "send to the other side" in the processes of solving an equation, when what was actually happening was the multiplication on both sides by the multiplicative inverse.

With respect to the combination of symbolic language and natural language to justify steps, there were different cases: where students tried to write the property using only natural language as in Figure 4, where they named it in natural language and then reinforced it with the symbolic language illustrated in Figure 5, or as shown in Figure 6 only with symbolic language.

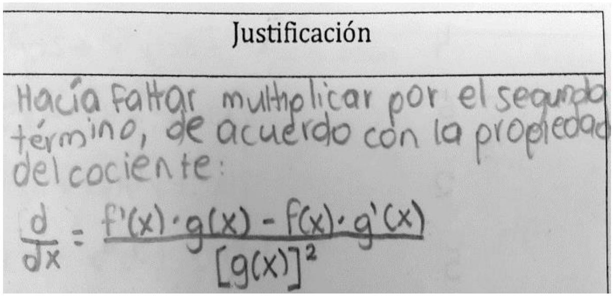


Justificación

se desarrolló la derivación de la división de dos funciones, por lo tanto primero se multiplica la función denominador por la derivada del denominador, que viene a ser una derivación de una multiplicación de funciones; a esto se le suma la derivada del denominado multiplicada por el numerador sin derivarlo. Todo se divide entre el denominador al cuadrado.

FIGURE 4. Example where only NL is used

The derivation of the division of two functions was developed; therefore, first the denominator function is multiplied by the derivative of the denominator, which becomes a derivation of a multiplication of functions; to this is added the derivative of the denominator multiplied by the numerator without deriving it. All that is divided by the denominator squared.



Justificación

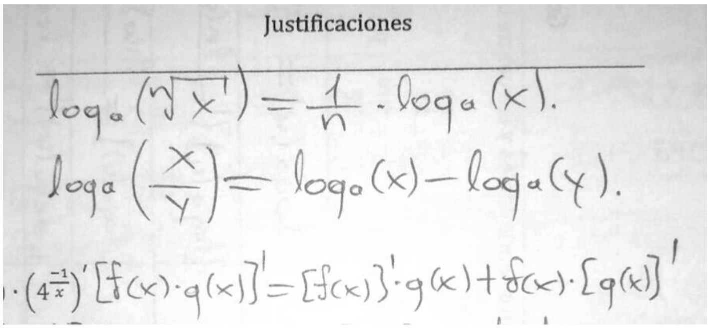
Hacía faltar multiplicar por el segundo término, de acuerdo con la propiedad del cociente:

$$\frac{d}{dx} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

FIGURE 5. Example where SL and NL are used to refer to property

It was necessary to multiply by the second term, according to the property of the quotient

$$\frac{d}{dx} = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{[g(x)]^2}$$



Justificaciones

$$\log_a(\sqrt[n]{x}) = \frac{1}{n}\cdot \log_a(x).$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y).$$

$$\cdot\left(4^{\frac{-1}{x}}\right)' \; [f(x)\cdot g(x)]' = [f(x)]'\cdot g(x) + f(x)\cdot [g(x)]'$$

FIGURE 6. Example where only SL was used

In Figure 4, we can see a description in words of the derivation rule for a quotient of functions, but in addition the student points out that in the numerator there is a product, which implies the application of the rule of the derivative of a product; whereas in the example of Figure 6, when using the formula in symbolic language, that is not evident.

It is important to mention that in the instructions, the students were asked to provide justifications using their own words. Even though they all had the same instruction, their responses varied in form, and not only by the type of language they used, as was shown in the previous examples, but also with respect to the length or detail of the justifications. While some students made clear justifications about the property or rule used, others simply pointed out the action that was being done, for example "derived," "simplified," or "solve."

The exercises made it possible to observe in some cases that the students mentioned why they were going to perform a step or procedure, that is, of the utility of doing it in that way. For example, in exercise 1, they identified that the algebraic operations developed were intended to give the limits the "shape" of the special trigonometric limits, and in exercise 9 they mentioned that logarithmic properties were used to make it easier to derive. This is important because it shows that students are not performing meaningless procedures, and that they recognize the utility of the tools and this helps them to identify when it is worthwhile to use them.

It was also possible to observe how the students justified a process using different mathematical knowledge. When in exercise 11 they had to justify why the identity $\cos y = \sqrt{1 - sin^2 y}$ was true, they mentioned the relation with the trigonometric circle, Pythagoras's theorem, the trigonometric identity and the use of a right triangle, and the trigonometric ratios. Interestingly in that same step, only one student justified the choice of the positive solution according to the domain of the inverse trigonometric function.

Group B

Exercises 5 (Appendix 6) and 8 (Appendix 9) require students to understand instructions given in natural language to develop procedures in symbolic language. In addition, in exercise 5 the instructions or steps are given in disorder. Exercise 5 presents the steps to solve a problem of tangent lines to a curve and exercise 8 is a proof of the derivation rule of logarithms in any base. In both exercises, the students were expected to use mainly symbolic language, since they had to perform calculations. However, some students included in their procedures natural language phrases that allowed them to order the procedures. For example, the words “then” and “if”, or phrases to indicate if they were calculating the slope of the line or of the curve, as shown in Figure 7.

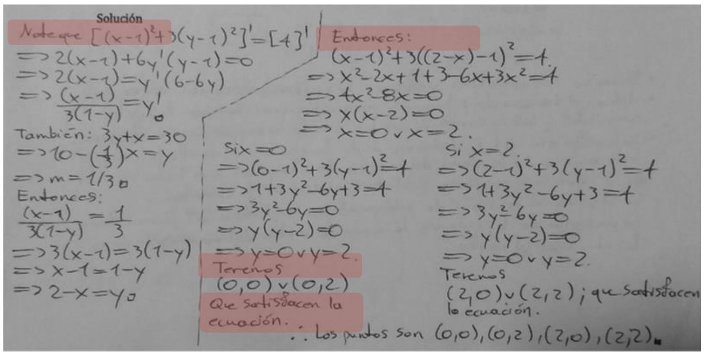


FIGURE 7. Example of use of phrases in NL

Most of the students ordered their procedures considering the steps given in the exercise, which made it easier for the reader to follow their reasoning. Likewise, the majority evidenced understanding what was indicated in each step, except the steps in which they were asked to verify that the ordered pairs were both, in the curve and in the line. This was evidenced since some got four ordered pairs or one only, when the correct answer was two.

It was also possible to observe that most of the students gave to the equation of the line the form $y = mx + b$, in order to identify the slope, even though they could have used the derivative. Interestingly, although they were studying the derivative, and that it can be interpreted as the slope of a line, the students prefer to use the equation of the line maybe because it is more familiar to them as they have been using it since secondary school.

Exercise 5 and 8 made possible to observe that the students did not monitor their work in detail, since they made many algebraic or sign errors, even several times within the same exercise, without realizing it. In the exercise 8, for instance, many of them could not reach the answer because they did not recognize that when changing the base of the logarithm in $\log_a x = \frac{\ln x}{\ln a}$, the expression $\ln a$ was a constant and its derivative was zero, and others used in the proof, the rule that they had to prove.

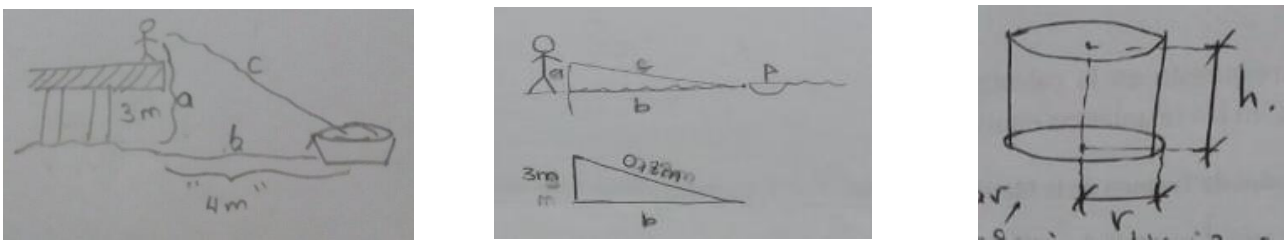
Group C

In the exercises of this group, students were asked to provide an explanation of how to solve the problems offered, in a way that can be understood by any of the classmates. In exercises 7 (Appendix 8) and 16 (Appendix 17) they could combine languages if they considered it necessary, while in exercise 2 (Appendix 3) they were required to use only natural language.

From the answers different types of explanations can be identified. The first difference is that some students wrote their explanations using a list of steps or notes and others offered them in the form of prose. Another difference is with respect to the depth of the explanations. Some merely stated the actions to be taken: "derivate" or "calculate the limit," while others explained in detail how to perform the procedures and why. This was possible to observe in exercise 7, where students had to add the explanation to the calculations given to solve a problem of rates of change, and in exercise 16, where they should explain how to solve an optimization problem.

Some students offered general explanations that could be used to solve any related problems, while others identified the particular formulas, variables, and processes for the given problem. It is important to mention that providing a general explanation may mean that they have been able to identify patterns to solve a certain type of problem, but it may also be associated with the fact that some textbooks, and even often within the teachers' discourse, a series of convenient steps to solve problems are offered.

The answers of exercises 7 and 16 involved all the three different languages. Some students included symbolic language by means of calculations to accompany explanations in natural language. However, almost all the students drew or suggested to make a drawing to be able to relate the data with the situation presented. The drawings A and B, in Figure 8, were presented in exercise 7, and it is possible to observe how the students were locating the variables and the values they were given to them. In drawing A, we can notice that there is a four in quotes, and it could be related to the fact that leg b takes measure 4 at a certain moment, it is not constant as is the case of leg a. Therefore, the drawings made possible to see if the student interpreted the statement correctly.



A B C

FIGURE 8. Examples of drawings

The most detailed explanations revealed that the students had knowledge of the theory and were able to understand even the specifics of the proposed exercise, some of them are presented in the following extracts.

To solve this problem, it is first necessary to understand that the rope, the surface of the water, and the height of the hands of the man with respect to the boat, form a triangle. The height of the man's hands remains constant (3m). First, it is important to find out the value of the side c *(hypotenuse) of the triangle, when the boat is 4m from the dock. Pythagoras is applied and 2 answers are obtained, and the positive one is chosen since there are no negative lengths. Then proceed to derive from both sides of equation the formula of Pythagoras, since it involves all the necessary values. It is important to remember that the derivative of "*c*" in the formula, is* $\frac{dc}{dt} = -0{,}8\,{}^{m}/_{s}$*. Therefore, when carrying out the operation, and making the necessary replacements, it will be possible to solve for b′, or the speed at which the boat approaches the dock (which will be negative because the leg b decreases in size).* *(S33)*

- *Make a representative drawing.*
- *Determine the variables that will be used in the problem.*
- *Determine what is necessary to optimize, depending on the problem, in this case it is necessary to optimize the total area.*
- *Propose an auxiliary equation to solve one variable in terms of the other, so that the total area is defined in terms of a single variable to facilitate the derivation of the function; in this case, solve the height in terms of the radius with the volume equation:* $V = \pi r^2 h \Rightarrow \frac{3}{2} = \pi r^2 h \Rightarrow \frac{3}{2\pi r^2} = h$
- *Replace the previous data in the formula of the total area and simplify, that is the function to be optimized.*
- *Determine the feasibility domain considering when the radius tends to 0 or towards infinity.*
- *Derive the criterion of the function of the area to be able to determine the relative maximums or minimums that will solve the problem.*
- *Determine the critical numbers and construct the sign table to know the relative minimum point of the derivative.*

- *Calculate the height of the cylinder with the value of r obtained as relative minimum.*
- *The values obtained are those that satisfy the required conditions.*

(S32)

These examples of explanations given by the students, demonstrate that they are aware of the theory involved in solving problems, making several connections between concepts and procedures. Nevertheless, exercise 2, in which students were to explain how to find the value of two constants so that a piecewise function could be derived at one point, an exercise that is usually solved by performing calculations, evidenced that the students were not clear about the process they should follow.

Many of the answers included only the analysis of the continuity or only the analysis of the derivate, although with this information it was not possible to determine both variables. Of the students who mentioned that it was necessary to analyze both derivability and continuity, very few explained how to do so. The next quotation is an example of an answer given to exercise 2, where the student mentioned the need of the two requirements, but it is not clear if he knows the correct way to prove them.

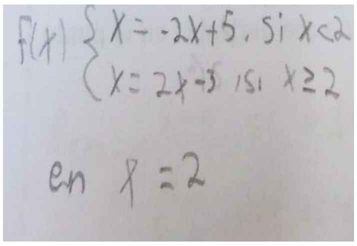
- *Analyse continuity at $x = 0$*
- *From this analysis obtain the value of a and b*
- *Study the lateral derivatives of the function, with the values of a and b obtained, to verify that these [the lateral derivatives]are equal*
- *If f is continuous and the lateral derivatives are equal then f is derivable at $x = 0$* *(S32)*

The types of exercises that involve explanations allowed to observe if students have an idea of how the theory is applied in different situations, as well as their ability to express and understand the processes they perform in order to obtain solutions.

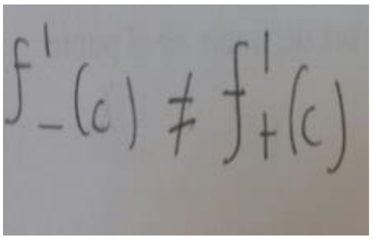
Group D

This group consists only of exercise 3 (Appendix 4), which has the particularity of requesting students the use of the three languages: symbolic, natural and pictorial, to provide examples of cases in which a function is not derivable. They were already given three cases where the function was not derivable, and each case had an example in one of the languages.

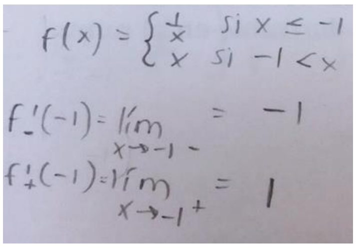
For the first case, it was given the affirmation in natural language: "At points where the curve has sharp points, since the lateral derivatives would be different." This statement does not refer to a particular function and makes a suggestion to the graphical form (sharp points) of the function where the derivability requirement is violated, as well as the theoretical aspect that fails (the lateral derivatives are different). This allowed the examples of the students, in symbolic language, to include equations of particular functions as in the example A of figure 9, or an expression in which the lateral derivatives were indicated to be different, shown in example B. Likewise, some students gave examples of a particular function and verified that the lateral derivatives were different, as the example C.



A



B



C

FIGURE 9. Examples for the first case, in symbolic language

Many of the examples given by the students in pictorial language, were related to the equation of the function chosen in the symbolic language example, and those who offered a more general response in symbolic language represented what was indicated in the given statement: a graph with a sharp point.

For the second case, students were given the expression $f(x) = \sqrt[3]{x},\ in\ x = 0$, which corresponds to an example where the derivative does not exist since the tangent line at that point is vertical. In this case, despite the fact that most of the students succeeded in sketching the graph of the given function, the answers in natural language varied considerably, as it is presented in the following examples.

- *In the points where there are vertical lines, because these have no slope and consequently also have no derivative. (S19)*
- *The derivate has a vertical tangent at that point. (S18)*
- *At points where the function is a vertical line (S17)*
- *When result of the limit is* $\frac{x}{0}$, *what is a vertical function (S27)*
- *There is a vertical asymptote. (S26)*

- *At points where the derivative has a $+\infty$, since this would mean a perpendicular tangent line, which does not exist. (S24)*
- *In points where the tangent line is vertical so that its slope is indefinite. (S30)*

These expressions show that although the students may have the right notion, that is to say that at that point the tangent line is vertical, they did not know how to express it correctly.

In the last case, a graphic of discontinuous function was given, and the task of the students was to provide the example in symbolic and natural language. Unlike the previous case, most of the students were able to correctly express the reason why it was not possible to derive the point indicated in the graph, offering theoretical foundations like "At points where the function is not continuous, because if it is not continuous at such a point, it cannot be derivable" (S24) or "at points where there is inevitable discontinuity or the general limit does not exist" (S30).

With respect to the examples in symbolic language, the situation of case one was repeated. Some students wrote general expressions where they indicated that the lateral limits were different, some tried to interpret the equation of the represented graph and others offered examples of functions defined by parts that had discontinuity at some point.

Group E

This last group includes exercises 12, 13, 14 and 15. In these exercises students were asked to justify or explain given statements. The answers may be accompanied by long paragraphs, a graphic or expressions in symbolic language. Since the basic knowledge involved in these exercises have already been studied in the previous classes, it was possible to observe how the students made connections or different applications of those contents.

In exercise 12 (Appendix 13), the students were faced with a situation in which they had to evaluate between two suggestions offered to find minimum points of a function, but before they had to describe the relation of the derivative with those points.

Among the answers to the first question, we can identify two main ideas, first that the minimum points are where the derivative becomes zero and second that they happen when the derivative changes from negative to positive. Very few students included the case where the derivative does not exist. This is exactly the same mistake made by Carlos in the statement of the exercise. However, despite student did not mentioned at the beginning the points where the derivative became zero as possible minimum; at the time of evaluating the opinions of Carlos and Lina in the statement, students identified those points as minimum and that Carlos was in a mistake. This could mean that the students did not return to evaluate their own answers and that probably if they had had to solve the problem without knowing the opinions of Carlos and Lina, their answers would have coincided with Carlos'.

By asking the students to justify for what certain procedure was necessary to perform in the solution of an exercise, it was possible to identify different opinions for the same procedure. For example, they comment that it is necessary to check the continuity to: find maximum and minimum points in a range, to be able to derive, to be able to apply the theorem of the extreme values or for checking that the image of the extreme point exist. All of these answers are correct.

From exercise 14 (Appendix 15), which included the knowledge to perform the analysis of a graph and to be able to graph, and exercise 15 (Appendix 16) in which the students should also interpret aspects for the analysis of a function, but from the graph of the first derivate, it was possible to observe that most of the students had the theoretical knowledge to solve that kind of exercises and that they were able to perform and express correct interpretations, even though these exercises were not what they were accustomed to do for those contents.

5.2 Students' perception of languaging exercises

The students' perception of the languaging exercises was valued from the answers of the questionnaire (Appendix 19). The results are presented in two parts. First, quantitative interpretations were done considering the responses of the Likert scale and second, with the answers to the open-ended questions, a content analysis was carried out from which five categories were obtained: students' opinion regarding lesson development, difficulties faced, benefits of the languaging exercises, possible uses of languaging and the general experience. The results will be described in the next sections.

5.2.1 Likert scale statements

The statements of the Likert scale assess the level of agreement of the students regarding: their abilities in mathematics, the utility of using the different languages in solving exercises and studying, as well as how difficult it is for them to use the languages.

From the chart shown in the appendix 21, we can observe that the students who participated in the study have a very high perception of their mathematical skills. 97% (N=33) of them agreed that they are good in math, and all the respondents think they are good at solving mathematics exercises.

73% of the students (N=33) considered themselves to be good at even solving the most difficult math exercises and only 24% (N=33) thought math is easier for others than for themselves. Those results are probably related to the fact that the participants are students of engineering major, and there is the belief that in order to be in that major you need to be good at math.

Regarding the uses of different languages when solving exercises and studying, the results showed that most of the students like the verbal tasks and that they think those tasks are useful. 94% (N=33) agreed that their written comments helped them to understand what was done in the exercises, to control the solution process (73%, N=33) and the exercises that use natural or pictorial languages, are easier to understand than those that only use symbolic language (85%, N=33). Although students liked verbal tasks, it is important to note that only 48% of them agreed that it is easy to write justifications.

About the symbolic language the students reported that they like to use it to justify their solution (64%, N=33), possibly because it is the one they are accustomed to use, while only 46% (N=33) believe that writing their thoughts in symbolic language is one of the most difficult things to do in mathematics.

In Table 4, we can observe some measures of central tendency that reflect the behavior of the answers.

TABLE 4. Results of the Likert scale

Statement	% Agreed	Mean	Standard Deviation
1	70	2,3	0,53
2	100	2,4	0,50
3	73	1,8	0,55
4	24	0,9	0,83
5	64	1,7	0,82
6	82	2,1	0,83
7	70	1,9	1,03
8	67	2,0	1,05
9	48	1,5	1,09
10	73	1,9	0,82
11	46	1,5	0,91
12	67	1,9	1,14
13	64	1,7	0,88
14	85	2,2	0,71
15	91	2,3	0,63
16	73	2,1	0,86
17	70	1,9	0,72
18	94	2,5	0,62

5.2.2 Open-ended questions

Based on the content analysis carried out with the answers to the open-ended questions, I organized the data in 5 categories based on the opinions of the students on the languaging exercises. The categories, the tables with students' comments and the frequency with which they were mentioned, and the respective analysis are presented in the following paragraphs.

I Lesson development: characteristics that the class acquires by including languaging exercises, including the work of the teacher.

TABLE 5. Students' comments about the lesson development

Students' phrases	***f***	**Students' phrases**	***f***
More dynamic class	1	Some of my classmates could not perform the exercises	1
Useful to exemplify problems	1	Design the exercises in an easy and didactic way	1
The student attention is better	1	It allows to follow the thread of the class, since the mathematical panorama is clear	1
Make the class lighter	2	The assessor can verify that the student understood what he is doing, and not only performing mechanical processes	1
Make the class heavier	1	We all had the same doubts	1
The teacher uses it a lot	1	Let the teacher explain himself better	1

Considering the answers to the open-ended questions, the students did not recurrently mention topics that are related to the exercises of languaging with the dynamics of the class. However, there are a few comments about how classes could be easier to follow, with the use of the languaging exercises and that the students paid more attention.

Besides, students mentioned that the teacher used the languaging a lot, which allowed him to express himself better and present the exercises in a more didactic way, as indicated in the following quotation:

“The teacher explains the exercises in an oral and pictorial way and also uses mathematical language. This mixture of languages helps a lot with the comprehension of the exercises, because if the concept explained in mathematical language is not clear, it will be clear with the explanation that involves images and "natural" words.” *(S22)*

The exercises also helped the teacher to verify that the students had really understood the subject and they were not only performing mechanical processes. This last action of the teacher, is related to the statement 16 of the questions of the Likert scale: “For the teacher, it is easier to evaluate that kind of exercises involving natural language and comments because it is easy to follow how the solver has understood the solution process,” which 73% (N=33) of the students agreed.

Some, however, noted that the exercises made the class heavier and that many of the classmates could not solve them.

II Difficulties: complications faced by students in solving the exercises, regarding their own limitations or due to the characteristics of the languaging exercises.

TABLE 6. Students’ comments about the difficulties faced

Students’ thoughts	*f*	Students’ thoughts	*f*
Very long	1	I am lazy to write, it's very tedious	2
Complex	5	They did not help me more than other exercises	1
Require more time	8	The exercises tend to confuse me	1
Less aesthetic	1	The annotations related to particular exercises are not useful for the exam	1
Asking for justification may not be the best	1	Make the solution of the operation more difficult	1
Justification is not so easy for me	1	I was not familiar with the type of exercise; we were never taught that	11
Translating procedures to words is difficult	1	It was difficult to explain at the beginning and follow the given steps of others	3

Translation of symbolic to natural is difficult	1	When there are only formulas, I may not understand what I do and just follow the formulas	1
The code language is more difficult to understand than the language of oneself	1		

Regarding the difficulties presented, the students indicated that the fact of not being familiar with the type of exercises complicated the solution process. For them, writing justifications and explanations is not a usual or easy task. This is supported by the numbers in the questionnaire, where only 48% (N=33) of the students agreed that writing justifications was easy for them and the following excerpt:

> *"In general we do not have the facility to express with words the procedures used, because we have never been taught that."* *(S11)*

In addition, they indicate that the translation between languages was also difficult for them, and that following and explaining steps or processes not made by themselves, was not easy. It could be because the action of understanding what other student did in the solution process implicates trying to understand how the other thought and what rules or properties used, so it requires knowledge of the content involved and different forms of solution.

There were also opinions regarding the structure of the languaging exercises. For example, students mentioned that languaging exercises were more complex, longer and required more time than others, and that writing was tedious and some of them were lazy to do it. However, these opinions were not supported by the majority, which is evidenced in the affirmation of the questionnaire "I am willing to use a long time for solving mathematics exercises," which 91% (N=33) of the participants agreed.

III Benefits of using different languages: includes the benefits mentioned by students for each language in particular and for learning at a general level. The benefits for learning are divided into benefits for interpreting and understanding, analyzing and reasoning and finally explaining or justifying.

TABLE 7. Students' comments about the languages benefits

	Students' thoughts	*f*	Students' thoughts	*f*
PL	Allows to visualize and imagine what happens in the problem, see it in a more real way	15	Images help avoid confusion	1
	Give more visual information and makes the data clearer	3	Allows to represent, demonstrate or clarify a problem	2
NL	Better and easier understanding speaking or explaining, than symbolically	5	Using natural language implies it will be understandable for the whole class	1
	It simplifies the process of understanding the exercises, because the numbers are removed	3	Method of understanding is more familiar to the ear	1
	It is easier to understand math with natural language, and is more applied to reality	2	Eliminates the need to learn so much mathematical symbols to justify the solutions	1
	It is more natural to explain something with words than with symbols	2	Express problems I could not solve, on words	2
SL	I consider myself better using symbols than languages, it is easier	4	Makes the answer shorter	1

According to the students' responses, one of the greatest benefits of using pictorial language in the statement or resolution of the exercises is that it allows to visualize, imagine and represent what the problem is about in order to understand what needs to be done. In addition, it is a way of presenting the data in a more obvious way, which helps to avoid confusion.

The use of this language was pointed out by some students as mandatory in the resolution of optimization exercises and related rates of change, to facilitate the understanding of the problem.

> *"It is very useful, especially in problems where it is important to visualize what is happening, the variables that we have and what we need to find."* *(S26)*

On the other hand, they consider the graphs as a manifestation of this type of language, and consider them difficult to interpret.

In general, the students stated that natural language simplifies and improves the process of understanding exercises, since it is more "natural," more familiar to the ear and easier for everyone to understand, especially for those who have difficulties with symbolic language. This is because, according to the students, it helps to understand the symbols or remove them from the way.

> *"I think it is easier to understand in general terms; it seems less aesthetic and demands more time (...) But I keep in mind that using natural language allows the explanation to be understandable for the whole class."* *(S12)*

It is important to mention that some students indicated that, for them, it was more useful and easier to use the natural language in speaking than in writing. This can be related with the fact that they are not use to writing justifications or explanations, mentioned in the category of difficulties.

> *"Personally, I use it, not written, but spoken, since it is easier for me to understand if I speak while I answer the exercises or try to solve them."* *(S24)*

> *"Having to write natural language is tedious, but using it (orally) is much simpler. The benefits are broad; most mathematical exercises are easier to understand when they are explained in an oral manner than only in mathematical language." (S32)*

The comments of the students coincide with the questionnaire since 85% (n=33) of the interviewees agreed that the mathematical exercises in which there are steps explained by natural language, are easier to understand than the one with only mathematic symbolic language. To these benefits is added that writing with words helps control the steps of a solution, which 73% (N=33) of students said is facilitated with these exercises.

Symbolic language was mentioned by a few students in open-ended questions. They indicated that it is easier and faster to use, and that it makes the answers shorter. It is important to note that in the questionnaire, 64% (N=33) of the students reported that they liked to justify their answers with symbolic mathematical language. On the other hand, only 46% (N=33) thought that in math the most difficult is to write your thoughts in mathematical symbolic form.

There were comments that did not refer specifically to some language, but mentioned benefits with respect to learning. These comments were divided into three categories: interpreting and understanding, analyzing and reasoning, and explaining and justifying.

TABLE 8. Students´ comments of the benefits of languaging

	Students' thoughts	*f*	Students' thoughts	*f*
Interpreting and Understanding	Facilitating the learning process making it more efficient	2	They serve to clarify doubts or gaps in the learning	3
	Is useful to see the steps when learning a new methodology	2	It is easier to understand the exercises both the statement and the procedures	23
	Allows to understand what I am doing or what I have to do	4	Helps to understand easier and better the contents	26
	See the structure improves understanding	1	To check something in an exercise	1
Analysis and Reasoning	Think about the purpose of each step	3	Being aware of my own understanding	5
	Know that what we do, makes sense	1	Develop critical and logical thinking, reasoning, and the ability to express ideas	4
	Order my ideas, thoughts and the solution of the exercise	7	Help to find errors	1
	Help me think in different ways	1	Allow to intuit properties, relations between concepts or algorithms, instead of memorizing them	3
Explanations and Justifications	We learned to explain better and in different ways the problems and the answers	7	To give an explanation implies a correct understanding of the contents	1
	Help to put mathematical reasoning into words, i.e. to build a verbal way of a solution	2	By justifying and giving explanations I understand what I do and why	2

The most mentioned topic was the usefulness of languaging exercises to improve or make easier the understanding and comprehension of contents, exercises, procedures and explanations. In other words, to understand what to do and how to do it, in a deeper way, as supported in the following quotation.

> *"It benefits the student because it helps them to understand what he is doing or has to do, in order to solve the exercise in the best way, according to the interpretation given."* *(S24)*

Besides, it is shown in the questionnaire that 70 % (N=33) of the students think that if their classmates use pictorial and natural language in the solution of the exercises, it is easy for them to follow the process.

The comments referring to more analytical actions are about understanding the reasons and justifications of the processes. In addition, they mentioned the development of logical and critical thinking, and the ability of reasoning and expressing thoughts. These abilities help students to be aware of the acquired learning, understand and express it, as well as give them more understanding to sort out their thoughts and procedures when solving an exercise.

> *"Stimulates logical thinking and reasoning. So one not only does it because an algorithm is memorized but by justifying it, one understands what and why one is doing it, and that helps in solving problems."* *(S30)*

The students commented that with the languaging exercises they learned to explain their procedures better and justify them, and that in order to give a good explanation they had to have a good understanding of the contents, because *"Coming up with an explanation implies a correct understanding of the subject." (S14)*

IV Possible Uses: related to the student's application of languaging as such or languaging exercises.

TABLE 9. Students´ comments about the possible uses of languaging

Students' thoughts	*f*	Students' thoughts	*f*
I will share the information with others	1	To study alone or with friends	6
I will use it to make notes, study sheets, summaries to study, and understand them	6	To solve the practices, to practice for the exam	7
To develop and express the problems solved in class and outside	1	Take verbal notes from the teacher's explanations	1
To explain to others	8	Could be used in other courses	1
When I study it is easier to understand what I do, having the explanation	3	To remember how an exercise was solved in class	1

According to the answers, the students mentioned that they will use the languaging in the future in different situations: to take notes of the teacher's explanations about the contents or about the procedures of the exercises, and then understand them outside the class when studying. Others said that they would use it to make summaries or study sheets. They also pointed out that it was very useful to study, in groups or individually, and when explaining their solutions or the contents to a classmate.

> *"While studying, having the explanation of why one step is done and another not, helps to understand the problem and facilitates the resolution of it."* *(S21)*
>
> *"I would use it to practice and study for an exam and to explain to someone, if necessary."* *(S17)*

The questionnaire statements related to the use languaging show that 70% (N=33) of students like to explain their solutions to the exercises to others, 67% (N=33) agreed that their written comments and annotations help them to solve the math exercises and 94% (N=33) state that when studying, their written comments help them to understand the solution process faster.

V General Perceptions of the Experience: general descriptions of the experience of solving languaging exercises.

TABLE 10. Students' comments about the experience

Students' thoughts	*f*	Students' thoughts	*f*
I used it before without realizing it	1	I like to find errors and order steps	1
Pleasant and good experience	8	Useful and important	12
Translate math into different situations	1	Not all minds are equal, getting to an exercise in all possible ways is fundamental	1
Useful for students with learning difficulties, especially those who do not understand the symbols	6	Original, never solved exercises like those ones	3
		I usually do not use languaging	3
Interesting see different approaches and descriptions of the problem	8	I was able to solve most of the exercises	1

Despite the difficulties faced in solving languaging exercises, students rated the experience as good, pleasant, and interesting, as well as emphasized that the use of different languages is very useful and important, especially for the students with different learning needs.

> *"Great, not all minds think alike, and getting to an exercise in every possible way seems fundamental to me."* *(S23)*

Besides, the results of the questionnaire show 64% (N=33) of the students liked the languaging exercises and 82% (N=33) of them thought they are useful. They liked the fact of seeing different approaches and descriptions of mathematical exercises. However, some commented that the exercises are not necessary and that they would not use the languaging in other occasions.

5.3 Teachers´ perception of languaging exercises

The interviews (Appendix 20) carried out with the teachers allowed to reveal the advantages and disadvantages of the exercises of languaging for both students and their learning, as well as for teachers and their teaching work. Next there are the main topics discussed by the two teachers, separated into benefits for learning and teaching, and the disadvantages.

5.3.1 Benefits for learning

The two interviewed teachers, P1 and P2 had different impressions of the languaging exercises. P1 focused his comments on the importance of languaging to attend to the individual differences of learning, since when using different languages, the student is offered the opportunity to make different mental representations of certain concepts or processes. According to their preference, the student can choose the representation in the language that is easier for he/she to understand and remember. In addition, this can lead to a more meaningful learning, since the different representations can facilitate connections between concepts. Furthermore, the student has different options to approach the problems and practice the ability to choose the most convenient representation according to the situation.

> *"(...) the fact that one gives students different possibilities to make the mental representation of a concept will generate some benefit because, if they do not achieve it through formal language, symbolic language, they will probably achieve it through the verbal language of themselves or their classmates or the same teacher. That is, it is simply that they are exposed to a different way of accessing the knowledge and assimilating it. (P1)*

On the other hand, P2 stressed that languaging exercises allow students to reflect more on the theory by having to consult and use it to make the justifications and explanations. This allows the students to question their learning and to see the applications of the theory in a more explicit way. She also mentioned that the exercises are useful to reinforce the contents studied in class and are a non-mechanical way of studying those contents.

> *"For example, there are some exercises that asked them to explain in their own words what is happening at each step. That seems important to me because it makes them reflect on the theoretical part, what the theory that we are using in each one*

> *of the steps of the exercise is. So it seems to me that helps them understand better and not make it so mechanical." (P2)*

As it is observed, the comments of P1 come from the consideration of languaging as a multimodal approach that reinforces the multiple representations; however, P2 was more focused on the usefulness of languaging exercises.

Regarding the skills developed by languaging exercises, the teachers emphasized the capacity of analysis, reasoning and abstract thinking, as well as some metacognitive skills such as being aware of their mental processes and their learning, and being capable of ordering their thoughts.

> *"metacognitive capacities many, (...) [the students] become aware of the mental processes that lead to solving certain types of exercises." (P1)*

In addition, they emphasize that the languaging exercises are useful to develop problem solving skills, since the students practice the interpretation of statements in different languages, they learn to monitor their work, to identify errors, to apply methodologies to different situations and to generalize processes.

Teachers also point out that the exercises addressed some very common problems of the students. For them it is often difficult to interpret statements in natural language, and identify mathematical models in symbolic language (i.e. equations) from them, as is the case of optimization problems. As well, they mention that the structure of languaging exercises, in words of P2 "very guided," helps students when studying because they can understand easily what was done and solve the exercises parallel with the theory.

5.3.2 Benefits for teaching

On the subject of using languaging for teaching, both teachers stated that they used it, somehow, in an orally way. P1 mentioned that it was part of his teaching style, because he always tries to represent the concepts using different mechanisms; however, he does not use it in a written way. He indicated that he usually verbalizes the procedures or definitions presented in with symbolic language on the board and that constantly asks the students questions in order to make them think about the theory, same comment of P2. Also, P1 mentioned the fact that there are concepts of topics that require the use of different languages to be explained clearly, as is the case of some optimizations problems and graph analysis.

Although the teachers already had used the languaging orally, both commented that the use of written languaging exercises, seemed to them novel and quite useful. P2, for example, commented that through the exercises she could tell what her students were thinking and the way in which they were doing it. In this way she could see if the students were understanding correctly or if they had misconceptions. She mentioned that the languaging exercises provided information to the teacher to modify her methodologies or teaching strategies.

> *"So I think it was a good and enriching experience for them and for me as a teacher, because those too are inputs that allow me to reorient the teaching. Suddenly a student arrives there with certain doubts and I say, well why he is asking me this, it should be that it is not clear such a thing of theory, (...) with the exercises, with the doubts they bring I can reformulate class orientation or reinforce such a thing." (P2)*

For his part, P1 comments that the experience with languaging exercises invites him to question whether, as a teacher, he makes an adequate use of the different languages, without favoring one more than others, so that no students will be at disadvantage. The teacher emphasizes the fact that historically, mathematicians have considered symbolic language as the only valid language in mathematics, for exercises and demonstrations; and that mathematics teachers sometimes repeat these tendencies.

> *"It was an interesting experience because, first as a teacher, it makes me to question my own strategies, to ask myself how much I am incorporating one or another type of language? Is it that I am very tight to one language and not incorporating others? (...), is it that I sometimes do not emphasize enough verbal language? do I rather demonize it? (P1)*

P1 also commented to be surprised by the expression skills of his students, who were able to provide good explanations for the exercises in natural language, although they were not accustomed to do it. This reaffirms the idea of individual learning differences, since it shows that it is easier for some students to perform processes using verbal language than perhaps in symbolic language.

> *"One could discover that the student has the ability to represent in words a process very well." (P1)*

5.3.3 Disadvantages

The main disadvantage mentioned by the teachers of applying the languaging exercises was the time, since the course is loaded with contents and that therefore they must cover them very fast and almost do not have time to dedicate the desired time to work with the exercises.

P1 stated the importance of discussing the exercises in the class in order to take advantage of them and avoid misunderstandings, for example in the exercises of finding errors. Dedicate enough time in the class for reviewing and commenting on the exercises is also important to prevent students from making false generalizations or over-generalizations of some processes or series of steps used to solve a particular exercise, due to those do not apply in exactly the same way for all exercises of that type; therefore, it could lead to unwanted mechanical repetition of procedures.

Another observation of P1 was the fact that there may be a devaluation of the use of symbolic language, if the students want to solve everything using natural language.

6 IMPLICATIONS AND CONCLUSION

Considering the purpose and the research questions that motivated this work, as well as the results presented in the previous chapter, it is possible to reach some conclusions and implications regarding the perception of students and teachers about the languaging exercises, as well as the utility of the exercises to observe how students express their thoughts about the mathematical knowledge related to the derivate. I am going to refer to them by answering the research questions respectively.

6.1 Concluding remarks

The first research question refers to how the students express their thoughts related to the mathematics knowledge. From the analysis of the responses of the languaging exercises, it was possible to observe important aspects about the way in which they articulate their mathematical ideas, with the help of the different languages. For example, the different uses students gave to languages; the different ways they created, with their own words and expressions, to refer to rules or procedures; the different knowledge and concepts they access to solve the same situation, among others.

When working with different individuals, it is very likely that we will obtain different opinions and different thoughts, and that is something that the languaging exercises allowed us to demonstrate very well. There were different ways in which the students named properties or processes, which not only highlighted the diversity and the importance of considering the different ways in which students learn, commented by teacher P1 in his interview, but also allowed identifying that some of those phrases constructed by students, can lead them to make mistakes. Therefore, the languaging exercises also offer a tool for teachers to be aware of students' meaning making and work on correcting misconceptions.

The diversity was not only evidenced in the students' sentences, but also in the different mathematical knowledge, content or procedures, to which the students had recourse to perform the same exercise. This shows that students make different connections within mathematical concepts, which can lead to different procedures and justifications. This kind of actions should be encouraged.

Another important aspect revealed in the exercises is that the participating students made their explanations and justifications with different levels of depth. This may be linked to the fact that they are not familiar with the tasks of describing processes or solutions in their own words, as the students manifested in the questionnaire; to the students' expression abilities or to their levels of knowledge. In spite of the differences and difficulties, the important thing is to promote exercises in class that engage students to explain the way in which they understand or solve the exercises, since according to Kline and Ishii (2008) and Sillius et al. (2011), this improves understanding, and in the same way makes students to order and clarify their mental processes, for them and for others.

Some explanations given by the students also showed that they were understanding the procedures and the aim of them, not only performing them mechanically, as is commonly done (Artigue, 1995; Valverde & Näslund-Hadley, 2011; Kilpatrick et al., 2001; Programa Estado de la Educación, 2013). Nevertheless, it was similarly evidenced that despite that the procedural abilities are the most practiced in school, students still are not aware of monitoring their solution processes and are making many basic algebraic mistakes.

In the exercises where students were asked to provide explanations of how they would solve problems, for instance, optimization or rate of change problems, it was evidenced that some students explained the process in a very general way, as if they were memorizing a list of steps. This situation should call the attention of teachers, because although it is true that one of the desirable skills to promote in students is that they are able to identify procedures that allow solving problems of certain characteristics, as indicated by Kilpatrick et al. (2001) within the strand the adaptive reasoning; this action must be carried out by the student himself. After solving a series of similar problems, the pupil can be motivated to identify steps and procedures, which according to his own experience allows him/her to solve those problems, like his/her own list of steps. In this way the student is going to be aware of importance and meaning of each step.

Last but not least, the solutions offered examples that reinforce the importance of the combination of the different mathematical languages within the class, both in the explanations and in the exercises that the students carried out, since each language allows to demonstrate different properties and characteristics of mathematical objects and processes, and as stated by Dreher, Kuntze and Lerman (2016) "a single representation can only emphasize some properties of a corresponding mathematical object "(p.364).

The students' perception of the languaging exercises is considered in the second research question. The results presented in section 5.2 evidenced that students consider that the languaging exercises are important, useful and provide an interesting way to present different possibilities to approach the contents. The students considered that the exercises are useful for different purposes. For example, for understanding more easily and deeply the exercises and mathematical contents they were studying, as well as being aware of their knowledge, ordering their thoughts and improving their skills to explain their procedures. Besides, they mentioned that languaging is a good tool when studying, taking class notes and explaining to others.

However, there were also some difficulties while solving languaging exercises. Students stated they were not used to or trained to write justifications and explanations; therefore, it was difficult for them to solve the exercises. Moreover, the design of some exercises and what they asked for were new for the students, for instance ordering given steps, and that makes the solution process seem more complex.

The opinion of the teachers who applied the exercises answers the third research question. The teachers emphasized that languaging exercises are a novel and very useful tool to promote learning and certain skills in students. The abilities of analysis, abstract thinking, reasoning and metacognitive skills, are favored when facing students to activities such as those proposed in the languaging exercises. The combination of all of them leads to more meaningful learning.

Teachers mentioned that the use of different languages attend to the differences in learning, since students are exposed to different mental representations of the concepts, and they can appropriate what they consider most representative for them. Besides, the fact that the students have to write in their own words the justifications for some procedures, help them to notice the use of theory to support what they are doing.

Regarding the teaching experience, they commented that introducing the languaging exercises in their classes make them to question their practice, think if they were supporting the use of one language above others. Likewise, by means of the answers the teacher can notice the way in which the students are understanding the concepts and if they have some misconceptions, and think about ways to reorient their teaching.

However, the biggest challenge they faced, and face every day, is the fight against time. A curriculum loaded with content and the pressure to study them all before the exam, limits the space for new methodologies, such as the languaging exercises.

In summary, considering the three sources of information: the answers to the language exercises, the questionnaires of the students and the interviews of the teachers, it is possible to conclude that the use of the different languages in the exercises makes the learning of the students to be more significant, exposes the way in which students take ownership of knowledge, considers differences in learning, evidences the use of mathematical knowledge and serves as a tool for students to study and understand the procedures performed, inside and outside the classroom.

In a context where there is a gap in the knowledge and the skills of students are limited to the repetition of algorithms, a tool that promotes students to acquire metacognitive skills of how and what they are learning, and that enables them to express their mental processes, is of great importance.

6.2 Limitations and Recommendations

Due to the characteristics of a qualitative study, the results are not generalizable, and therefore the opinions on the usefulness of the exercises of languaging in the University of Costa Rica are limited to the group of participants. However, the acceptance and the positive opinions on the convenience of the languaging exercises for improving learning shown in the results, offer an encouraging picture.

Regarding the design of the study, there are some limitations. For example, it does not allow to verify if the performance of the students in the course improved after the application of the exercises; though, the performance is affected by many factors so it would be very difficult to prove how much the exercises influenced. Furthermore, due to the researcher and the teachers were in different countries at the beginning of the process, the communication with the teacher was difficult and it was not possible to involve them in the design of the exercises.

For further research, it can be valued to introduce to the teachers the theories of mathematical proficiency, multimodal approach and languaging, so that they can design the language exercises for their own classes, considering their teaching style and time management. Besides, it can be explored the use of the languaging exercises as a tool to evaluate knowledge, for example in the exams; as well as the instruction of the teachers for the interpretation of the answers to the languaging exercises, especially those that involve natural and pictorial languages.

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APPENDIXES

Appendix 1

Contents of Derivate in the program of the course Calculus I

3 - 7 April	Definition of derivate. Relationship between continuity and derivability. Rules of derivation of algebraic and trigonometric functions. Derivatives of higher order.
17 - 21 April	Implicit derivation. Tangent and normal line to a curve.
24 -28 April	The derivate as instantaneous rate of change. Problems of related rates.
	Derivative of the exponential function. Theorem of the derivative of the inverse function. Derivate of the logarithmic function.
1 - 5 May	Logarithmic derivation. Derivate of the inverse trigonometric functions.
8 - 12 de May	Calculation of limits by the rule of L'Hôpital, indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\cdot\infty, \infty^0, 0^0, 1^\infty$ Absolute and relative extremes. Critical point. Theorem of the extreme value.
15 -19 de May	Calculation of extreme values for a continuous function in a closed interval. Rolle's theorem and mean value theorem. Relationship between the monotony of a function and the sign of the first derivative. Relationship between the concavity of a function and the sign of the second derivative. Inflection point. Criterion of the first derivative. Criterion of the second derivative.
22 - 26 May	Complete study of a function given its criteria: domain, intersections with axes, asymptotes, critical points, classification of relative extremes, growth and decrement intervals, inflection points, concavity, summary and plot.
29 May - 2 June	Optimization problems.

1

Languaging Exercises

Dear student, the answers to the language exercises will be used for research at the University of Tampere, Finland. The information will be handled confidentially, the results could appear in a scientific publication or be disclosed in a scientific meeting but in an anonymous way. Your answers will not affect your assessment in the MA1001 course. The purpose of the research is not to judge whether the answers are correct or not, but rather to evaluate the utility of these types of exercises to access your mathematical thoughts, please be as honest as possible.

Here is the incomplete solution of an exercise. Complete the solution of the derivative and describe in your own words the procedures of each step and the rule or property that justifies them.

Exercise: Using the definition, calculate the derivative of $f(x) = \cos(x) + 1$

Solution: The derivative of f in x is defined as: $f'(x) = \boxed{}$

Then replacing $f(x) = \cos(x) + 1$ and $f(x + h) = \boxed{}$, we have

	Justifications
$f'(x) = \lim\limits_{h\to 0} \frac{[\cos(x+h)+1] - [\cos(x)+1]}{h}$	
$= \lim\limits_{h\to 0} \frac{\cos(x+h)+1-\cos(x)-1}{h}$	
$\boxed{}$	
$= \lim\limits_{h\to 0} \frac{\cos(x)[\cos(h)-1] - \text{sen}(x)\,\text{sen}(h)}{h}$	
$\boxed{}$	
$= \cos(x)\lim\limits_{h\to 0}\frac{\cos(h)-1}{h} - \text{sen}(x)\lim\limits_{h\to 0}\frac{\text{sen}(h)}{h}$	
$= \cos(x)\cdot 0 - \text{sen}(x)\cdot 1$	
$= -\,\text{sen}(x)$	

What could you conclude from the answer about the derivate of cos(x)? Explain in your own words.

2

Languaging Exercises

Student number:

---

a) Write in your own words how you interpret the following mathematical expressions.

- $\lim_{x \to -1} h(x) = 2$
- $f'(3) = -1$

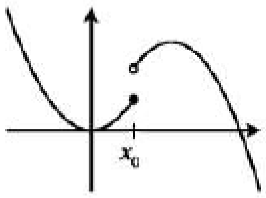
b) Explain in your own words, without using symbolic language, the steps you would take to find the value of a and b so that the function is derivable at the point x = 0. Justify your steps.

$$f(x) = \begin{cases} \sin(x), & if\ x \leq 0 \\ -x^2 + ax + b, & if\ x > 0 \end{cases}$$

Languaging Exercises 3

Student number:

What are the possible cases in which a function is not derivable? Give examples of each of them using the three types of language.

Mathematical symbolic: numbers, symbols.	**Natural Language: written words.**	**Pictorial Language: drawings, graphs, etc.**
	At points where the curve presents peaks, since the lateral derivatives would be different.	
$f(x) = \sqrt[3]{x}, in\ x = 0$		
		

Appendix 5: Languaging Exercise 4

Languaging Exercises 4

Student number:

The solution of the derivative of a mathematical expression is given below. Your task is to find the errors in the procedure, correct them and write in your own words the justifications for each step.

Solution	**Justificación**
$y = \frac{(2x+1)\cdot sen^2(x^3+1)}{\sqrt{3x^2+5}}$	
$y' = \frac{(2x+1)'\cdot sen^2(x^3+1) + (2x+1)[sen^2(x^3+1)]' - \left(\sqrt{3x^2+5}\right)'\cdot[(2x+1)\cdot sen^2(x^3+1)]}{\left(\sqrt{3x^2+5}\right)^2}$	
$y' = \frac{2\cdot sen^2(x^3+1) + (2x+1)\cdot 2\cdot sen(x^3+1)\cdot 3x - \frac{1}{2}(x^2+2)^{\frac{-3}{2}}\cdot(2x+1)\cdot sen^2(x^3+1)}{(3x^2+5)}$	

5

Languaging Exercises

Student number:

Order the following sentences to allow solving the exercise by placing the numbers in the boxes, then perform the calculations necessary to obtain the answer

- Given the curve defined by the equation $(x-1)^2 + 3(y-1)^2 = 4$ determine the points on that curve in which the tangent line is parallel to the line with equation $3y + x = 30$.
-

¿Do you think the equation $(x-1)^2 + 3(y-1)^2 = 4$ represents a function? Justify

Steps

	As we look for the points of the curve in which the tangent line is parallel to the line $3y + x = 30$, then the slopes must be equal.
	Select ordered pairs that satisfy both conditions.
	Identify the shape of ordered pairs that meet the condition of the slopes.
	Find the derivative of the curve equation $(x-1)^2 + 3(y-1)^2 = 4$, to know the shape of the slope.
	Clear one variable in terms of the other.
	Find the points that satisfy the condition of the slopes and at the same time are in the curve, substituting in the equation $(x-1)^2 + 3(y-1)^2 = 4$.
	Find the slope of the line $3y + x = 30$.

Solution

Appendix 7: Languaging Exercise 6

Languaging Exercises 6

Student number:

The following table shows the disordered steps of the solution of the derivative of the curve $(x^2+y^3)^2 = 4x^2y$, order the steps and comment in your own words which rule is applied or which procedure is carried out in each of them.

#	Steps	Justificaciones
	$2x^3 + 3x^2y^2y' + 2xy^3 + 3y^5y' = 4xy + 2x^2y'$	
	$y'(3x^2y^2 + 3y^5 - 2x^2) = 2(2xy - x^3 - xy^3)$	
	$2(x^2 + y^3)(2x + 3y^2y') = 4(2xy + x^2y')$	
	$3x^2y^2y' + 3y^5y' - 2x^2y' = 4xy - 2x^3 - 2xy^3$	
	$y' = \dfrac{2(2xy - x^3 - xy^3)}{(3x^2y^2 + 3y^5 - 2x^2)}$	
	$[(x^2 + y^3)^2]' = (4x^2y)'$	
	$(x^2 + y^3)(2x + 3y^2y') = 2(2xy + x^2y')$	

7

Languaging Exercises

Student number:

Here are the calculations of the solution of a problem of rate change. Explain and justify in your own words that solution, so that it can be understood by any student of the course. You can include any language you deem necessary for your explanation.

A man stands on a pier and pulls a boat through a rope. His hands are 3 m above the mooring of the boat. When the boat is 4 m from the pier the man is pulling the rope at a speed of 80 cm / s. How fast does the boat approach the pier?

Solution:

$a = 3$

$b' = ?$

$c' = -80\ ^{cm}/_{s} = -0{,}8\, m/s$

$$a^2 + b^2 = c^2$$

$when\ b = 4$

$$3^2 + 4^2 = c^2$$

$$\Leftrightarrow 9 + 16 = c^2$$

$$\Leftrightarrow \pm\sqrt{25} = c$$

$$\Leftrightarrow -5 = c \ \vee\ 5 = c$$

$$\therefore c = 5$$

$$(a^2)' + (b^2)' = (c^2)'$$

$$0 + 2b \cdot b' = 2c \cdot c'$$

$$2 \cdot 4 \cdot b' = 2 \cdot 5 \cdot -0{,}8$$

$$b' = -1$$

$$\therefore b' = -1\ ^{m}/_{s}$$

8

Languaging Exercises

Student number:

A) Consider the instructions given below to find the formula of the derivative of the logarithm on any basis.

Instructions

Knowing that $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot u'$, calculate $\frac{d}{dx}(\log_a u)$.

	Procedures
Apply the base change property: $\log_a x = \frac{\log_b x}{\log_b a}$ to a convenient base	
Derive on both sides, using the ratio rule.	
Use the formula $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot u'$	
Simplify	
Therefore the formula to derive a logarithm of any base is:	

Appendix 10: Languaging Exercise 9

Languaging Exercises

Student number:

9

Below is the incomplete solution of the derivative of a mathematical expression. Complete and explain in your own words what happens at each step. Some properties of the logarithms have been used.

Calculate the derivative of $y = \log_3 \sqrt[5]{\frac{x^2-1}{x^2+1}} \cdot 4^{\frac{-1}{x}}$

Justifications

$$y' = \frac{1}{5}(\log_3(x^2-1) - \log_3(x^2+1))' \cdot 4^{\frac{-1}{x}} + (\log_3(x^2-1) - \log_3(x^2+1)) \cdot \left(4^{\frac{-1}{x}}\right)'$$

$$y' = \frac{1}{5}\left[\frac{2x}{\ln 3} \cdot \left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right) \cdot 4^{\frac{-1}{x}} - \frac{4^{\frac{-1}{x}} \cdot \ln 4}{x^2} \cdot \left(\log_3 \frac{x^2-1}{x^2+1}\right)\right]$$

10

Languaging Exercises

Student number:

Be Below are the disordered steps of the solution of the derivative of a mathematical expression. Order them by placing the numbers in the boxes and explain in your own words the procedures or rules that apply at each step.

Calculate the derivative of $y = \left(\frac{\ln(2x+\sqrt{x})}{tan^2(x)}\right)^{\frac{1}{x^8}}$

	$y = \left(\frac{\ln(2x+\sqrt{x})}{tan^2(x)}\right)^{\frac{1}{x^3}}$
	$\frac{1}{y} \cdot y' = \frac{-3}{x^4} \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)] + \frac{1}{x^3} \cdot [\ln(\ln(2x+\sqrt{x}))' - \ln(tan^2x)']$
	$(\ln y)' = \left[\frac{1}{x^8} \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)]\right]'$
	$y = \left(\frac{\ln(2x+\sqrt{x})}{tan^2(x)}\right)^{\frac{1}{x^8}} \cdot \left\{\frac{-3}{x^4} \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)] + \frac{1}{x^3} \cdot \left[\frac{4\sqrt{x}+1}{\ln(2x+\sqrt{x})(2x+\sqrt{x}) \cdot 2\sqrt{x}} - \frac{2sec^2x}{\tan x}\right]\right\}$
	$\ln y = \frac{1}{x^8} \cdot \ln\left(\frac{\ln(2x+\sqrt{x})}{tan^2x}\right)$
	$\frac{1}{y} \cdot y' = \left(\frac{1}{x^3}\right)' \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)] + \frac{1}{x^3} \cdot [\ln(\ln(2x+\sqrt{x}) - \ln(tan^2x))]'$
	$\frac{1}{y} \cdot y' = \frac{-3}{x^4} \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)] + \frac{1}{x^3} \cdot \left[\frac{1}{\ln(2x+\sqrt{x})(2x+\sqrt{x})} \cdot \left(2 + \frac{1}{2\sqrt{x}}\right) - \frac{1}{tan^2x} \cdot 2\tan x\ sec^2x\right]$
	$\frac{1}{y} \cdot y' = \frac{-3}{x^4} \cdot [\ln(\ln(2x+\sqrt{x})) - \ln(tan^2x)] + \frac{1}{x^3} \cdot \left[\frac{1}{\ln(2x+\sqrt{x})} \cdot (\ln(2x+\sqrt{x}))' - \frac{1}{tan^2x} \cdot (tan^2x)'\right]$

Justifications

11

Languaging Exercises

Student number:

The formulas of the derivatives of the inverse trigonometric functions behave differently from those of the basic trigonometric functions. For example, they do not include trigonometric expressions but rather algebraic expressions. In order to know where this formula is obtained, the calculations necessary to proof one of the formulas are presented. Your task is to justify with your own words each step, complete the calculations (if necessary), so that the proof can be understood by any student in the course.

$Proof\ that\ \frac{d}{dx}(arcsen\ x) = \frac{1}{\sqrt{1-x^2}}$

	Justification
$y = arcsen\ x$	
$\Leftrightarrow sen\ y = x$	
$(sen\ y)' = (x)'$	
$\cos y \cdot y' = 1$	
$y' = \frac{1}{cosy}$	
$y' = \frac{1}{\sqrt{1 - sen^2 x}}$	
$y' = \frac{1}{\sqrt{1 - x^2}}$	
$\therefore (arcsen\ x)' = \frac{1}{\sqrt{1 - x^2}}$	

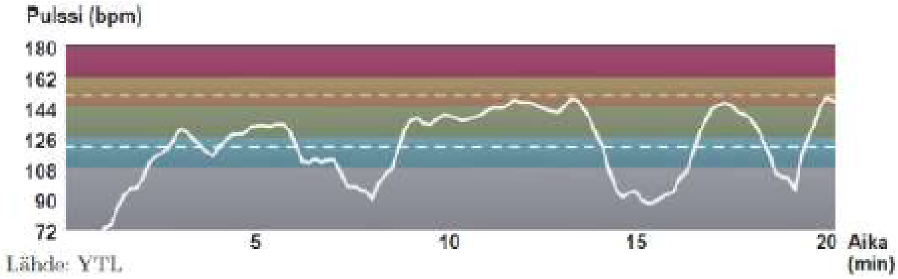
Languaging Exercises 12

Student number:

Kalle and Leena are doing the basics Biophysics course assignments. They have measured the subject's pulse $f\ (t)$, during sports activities at time t. The result is in the curve below. Measuring apparatus transfers the results in digital format in a wireless direct to Kalle and Leena's computer. Kalle and Leena's task is to program the computer to automatically calculate how many local minimum point of the pulse curve is. Kalle proposes a function to search for the zeros of the derivative. Leena says this sometimes works, such as the case of t = 15.2 min, but not always, e.g., when t = 19.3 min.

a) Describe in your own words, what is the relationship between the derivative and the minimum of the function.

b) Evaluate Kalle's proposal to find local minimum points, <u>as well as put forward by Leena's attention.</u>



Source: Finnish Board of Matriculation Examination, Finland, Spring, 2017.

Appendix 14: Languaging Exercise 13

13

Languaging Exercises

Student number:

Following are instructions for resolving the exercise. Your task is to explain in your own words why and for what it is necessary to follow these steps to find the answer. It is not necessary to solve it.

Given the function $g(x) = x^3 + 3x + 2$ in the interval $[-2, 3]$. Determine the absolute extreme values.

1. Verify that f is continuous in the closed interval.

2. Find the critical points of f.

3. Evaluate f on each of the critical numbers.

4. Evaluate f at the ends of the interval.

5. Classify the maximum and minimum.

Appendix 15: Languaging Exercise 14

14

Languaging Exercises

Student number:

For some function f the following information is given.

- $f\ is\ continuous$
- $f(-1) = -1,\ f(2) = -1,\ f(-3) = 4, f(0) = 0$
- $f'(-1) = 0, f'(2) = 0$
- $f'(x) = 0\ if\ x < -3$
- $f'(x) < 0\ in\ the\ intervals\ ]-3,-1[\ y\ ]0,2[$
- $f'(x) > 0\ in\ the\ intervals\ ]-1,0[\ y\ ]2,+\infty[$
- $f''(x) > 0\ in\ the\ intervals\ ]-3,0[\ y\ ]0,5[$
- $f'(x) < 0\ in\ the\ intervals\ ]5,+\infty[$
- $\lim_{x\to+\infty} f(x) = 6$

Explain in your own words the graphic implication of each of the above points. Then make an outline of the chart that meets the conditions.

15

Languaging Exercises

Student number:

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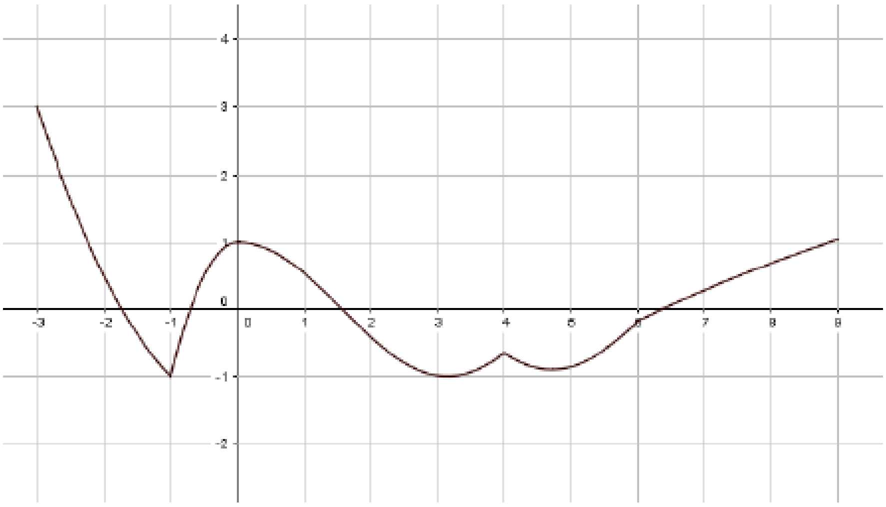
The figure below corresponds to the graph of the derivative of the function f in the interval [-3,9].



Justify in your own words the following statements based on the graph.

- x = 4 and x = 0 are points of inflection.
- The function f has two relative maximum points and two relative minimum points.
- The function f is concave upwards in the intervals] -1.0[,] 3,4 [and] 4.7,9[
- The function f is decreasing in the intervals]-1.7, -0.7[and]1.6,6.3[

16

Languaging Exercises

Student number:

Consider the following problem: A manufacturer needs to make a cylindrical can containing 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Suppose you should explain to one of your classmates how to solve this problem. Write in your own words what explanation you would give by justifying your affirmations, include the symbolic or pictorial elements you consider necessary. You do not need to solve the problem. Also explain why derivation is helpful for solving these problems.

Solution

Appendix 18: Languaging Exercise 17

Languaging Exercises 17

Student number:

We had $h(x) = g(f(x)), f(x) = e^x$ and $g(x) = 2x^2 + 1$. Henri and Antti derivate $h'(x)$ as follows:

Henri's answer	Antti's answer
$f(x) = e^x$ $g'(x) = 4x$ so $h'(x) = g'(f(x)) = 4e^x$	$h(x) = g(f(x)) = 2(e^x)^2 + 1 = 2e^{x^2} + 1$ $h'(x) = 2e^{x^2} \cdot (2x)$ so $h'(x) = 4xe^{x^2}$

Maja got the answer $4e^{2x}$ in the calculator. Who had the right answer? Find the errors in the wrong answers and present the correct solution.

Source: Finnish Board of Matriculation Examination, Finland, Spring, 2017.

Appendix 19

Questionnaire

Student number

Group

Answer to the following statements by placing a cross in a suitable box.

0 = Completely disagree 1 = Somewhat disagree

2= Somewhat agree 3= Completely agree

Question	0	1	2	3
1) I am good at Math				
2) I am good at solving math exercises				
3) I can even solve the most difficult math exercises				
4) Math is harder for me than for others				
5) I like verbal tasks				
6) Verbal tasks are useful in my opinion				
7) I like to explain to other students my solutions of the math exercises				
8) My written comments and subtopics help to solve math exercises				
9) I think it is easy to write the justifications				
10) Writing in words helps me to control the solution process				
11) In math the most difficult is to write your thoughts in mathematical symbolic form				
12) When I solve math tasks, I have the ideas in my mind and then write only what is necessary				
13) I like to justify my solutions with mathematics symbolic language				
14) The mathematical exercises in which there are steps explained by natural language, are easier to understand than the one in which there is only mathematic symbolic language				
15) I'm willing to use what? a long time for solving mathematics exercises				
16) For the teacher, it is easier to evaluate that kind of exercises involving natural language and comments because it is easy to follow how the solver has understood the solution process				
17) If my classmates use pictorial and natural solution, it is easy for me to follow the solution process				
18) When I am studying, my written comments help me to understand the solution process faster				

Open question: Give your opinion using your own words

What do you think about using natural language in the solutions of the mathematics exercises? What kind of benefits can you have from using it?

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What do you think about using pictorial language in the solutions of the mathematics exercises? What kind of benefits can you have from using it?

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What were your experiences about languaging exercises?

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How did your classmates experience languaging in studying mathematics?

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How would you use languaging in your studying at university?

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How did the languaging exercises help to the development of the class?

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Appendix 20

Guide questions of the semi-structured interview

Regarding Students' learning

Do you think languaging exercises help students to understand the mathematical concepts better? Why?

How do you think these exercises benefit students studying?

How do you think these exercises benefit students learning?

Regarding Teaching

Do you think this kind of exercises are common? Have you used them before?

Do you think the languaging exercises are useful for noticing and evaluating students' thoughts? Why?

What advantages can you find in using the languaging exercises for your teaching?

What disadvantages?

How would you use them in your classes?

Regarding languaging exercises characteristics

Which abilities do you think the languaging exercises promote?

Which model of languaging exercise do you think is more useful for promoting these abilities?

What changes would you do to the languaging exercises?

What was your experience with the languaging exercises?

Appendix 21

Students' Perception of languaging